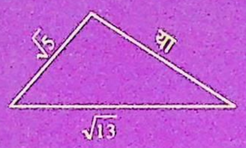
BĪJAPALLAVA OF KṚṢṆA DAIVAJÑA

ALGEBRA IN SIXTEENTH CENTURY INDIA

A CRITICAL STUDY

त्रयोदश तथा पञ्च करण्यौ भुजयोर्मिती ।
भूरज्ञाताऽत्र चत्वारः फलं भूमिं वदाऽऽशु मे ॥

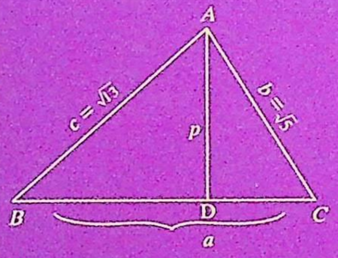


Projection $BD = \frac{1}{2}\left[a + \frac{(c+b)(c-b)}{a}\right]$

Projection $CD = \frac{1}{2}\left[a - \frac{(c+b)(c-b)}{a}\right]$

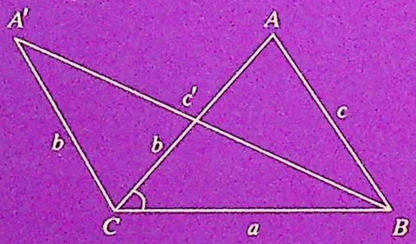
$p^2 = c^2 - BD^2$ or $b^2 - CD^2$

Area $\triangle\ ABC = \frac{1}{2}\ p \times a$



$$4 = \frac{1}{2}\ .\ \sqrt{13}\ .\ \sqrt{5}\ .\ \sin C$$

$$= \frac{1}{2}\ .\ \sqrt{13}\ .\ \sqrt{5}\ .\ \sin\,(180 - C)$$



Dr. SITA SUNDAR RAM

THE KUPPUSWAMI SASTRI RESEARCH INSTITUTE
Chennai - 600 004.

2012

"The importance of the *Bījapallava* has long been recognised but hitherto it has only been studied in parts that too, to a very limited extent. Dr. Sita's study is the first to treat the work comprehensively, in its entirety. It provides us with a clear image of Kṛṣṇa's Algebra dwelling on his merits as well as short-comings. . . . A researcher of the history of mathematics is not a treasure-hunter: he/she should be a geologist capable of unravelling the different historical layers, . . . Dr. Sita is qualified to be a geologist in the discipline of the History of mathematics "

– *Takao Hayashi*

"Falling short of an exhaustive translation of *Bījapallava,* the present book is a comprehensive critical study of the work. . . . Dr. Sita has done full justice to the work and has shown her mettle, as the contents of the book reveal"

– *M.S. Rangachari*

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Dr. V. Kameswari

Dr. K.S. Balasubramanian

Dr. T.V. Vasudeva

Dr. Sita Sundar Ram

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FOREWORD

It is indeed gratifying that the Kuppuswami Sastri Research Institute is publishing Dr. Sita's Ph.D. thesis, which is a pioneering study of Kṛṣṇa Daivajña's famous commentary *Bījapallava* on the *Bījagaṇita* of the celebrated mathematician-astronomer Bhāskarācārya.

Kṛṣṇa Daivajña belonged to a prominent *jyotiṣī* family of Vārāṇasī and was associated with the Mughal court at Agra. In the concluding verses of the *Bījapallava* (see §1.6.1 of the present work), Kṛṣṇa gives his genealogy where he praises the erudition of his father Ballāla, grandfather Trimalla, great grandfather Rāma and Rāma's father Cintāmaṇi. Among his five brothers, Raṅganātha is noted for his commentary *Gūḍhārthaprakāśaka* on the *Sūryasiddhānta*. Several *Jyotiṣa* works by his nephews Munīśvara, Gadādhara and Nārāyaṇa have come down to us (see D. Pingree, *Jyotiḥśāstra*, p. 126, for their family tree). According to Munīśvara, the family originally lived at Dadhigrāma on the Payoṣṇī river in the Vidarbha region, but Kṛṣṇa's father Ballāla moved to Vārāṇasī (*cf.* D. Pingree, *Census of the Exact Sciences in Sanskrit*, =CESS, A4, p. 432ab).

In the opening verses of the *Bījapallava*, Kṛṣṇa mentions his illustrious *guru paramparā* (see §1.6.2 of the present work). Kṛṣṇa's teacher Viṣṇu was the pupil of Nṛsimha (b. 1548) who was himself the pupil and nephew of Gaṇeśa (b. 1507), the celebrated author of the *Grahalāghava*, who studied

under his own father Keśava (fl. 1497/1507). This Viṣṇu is presumably the son of Divākara, the pupil of Gaṇeśa, who wrote *Sūryapakṣaśaraṇa* (see D. Pingree, *Jyotiḥśāstra*, p. 37). His works on mathematics are not known but his two rules for specific types of equations are cited by Kṛṣṇa in the *Bījapallava* (see §6.3 of the present work).

At the Mughal court, Kṛṣṇa received patronage from Jahāngīr as well as from Abd al-Rahim Khan-i Khanan. The horoscope of the latter was included by Kṛṣṇa in his commentary on Śrīpati's *Jātakapaddhati*. Kṛṣṇa was a member of the team entrusted with the task of rendering Ulugh Beg's astronomical tables into Sanskrit at Akbar's translation bureau. In a Mughal miniature painting depicting the birth of Jahangir, there is a picture of an astrologer drawing a horoscope. S.R. Sarma identifies this astrologer as Kṛṣṇa (see his *The Archaic and the Exotic*, pp. 100-07).

Aside from an original work *Chādakanirṇaya* on eclipses (*cf.* §1.6.4 of the present work), Kṛṣṇa produced commentaries on Bhāskarācārya's *Līlāvatī* and *Bījagaṇita* and on Śrīpati's *Jātaka*(*-karma-*)*paddhati*. The *Janipaddhati* on which Kṛṣṇa is said to have written a commentary (*vṛttī*) according to Munīśvara (*cf.* CESS A4, p. 438b) seems to me to be Śrīpati's *Jātakapaddhati*, although Dr. Sita regards it as a separate work.

David Pingree lists two manuscripts in the CESS with the comment, 'A ṭīkā on the *Līlāvatī* of Bhāskara II (b. 1114) is ascribed to Kṛṣṇa, but it is probably a confusion with the *Bījāṅkura* (CESS A2, p. 55b). However, there cannot be any doubt that Kṛṣṇa wrote a commentary on the *Līlāvatī*, for he himself refers to it in the following words: '*asti cācāryeṇa pāṭyām uktam*/ (verse 43 of the *Līlāvatī* is cited here) *iti*/ *upapāditam cāsmābhis*

tadvyākhyāvasare' (*Bījapallava*,Tanjore edition, p. 56; Ānandāśrama edition, p. 39).

The *Bījapallava* (also known as *Bījāṅkura*, *Navāṅkura* or *Kalpalatābatāra*) must have been composed before 1601 when its oldest datable manuscript was copied. It is a fullfledged commentary which chronologically comes next to Sūryadāsa's *Sūryaprakāśa* (1538). Generally speaking, commentaries, written in prose and therefore free from metrical constarints, supply us with more detailed information about the subject and this information often includes original contributions by the commentators which are as valuable as the contributions of the original authors. The *Bījapallava* also contains much that is original.

Under each *mūla* stanza or group of *mūla* stanzas, Kṛṣṇa first explains the words used there, giving their grammatical derivations, synonyms and syntactic combinations. More important, he cites sometimes variant readings and explains which of them is preferable. Then he goes on to discuss the mathematical contents in great detail, giving proofs (*upapattis*) for the rules and step-by-step solutions for the examples; but when the solution is easy, he merely refers to Bhāskara's auto-commentary. His discussions, often in the form of disputations between an imaginary opponent and himself, go deep into the nature of important mathematical concepts such as negative quantity, zero and unknown quantity, into the *raison d'être* of particular steps of the algorithms, and into various conditions for solubility of the mathematical problems treated in the *Bījagaṇita*. He even provides his own rule with two examples for a certain type of problem at the end of Chapter 9 which deals with equations in more than one unknown number (see §6.2 of the present work).

The importance of the *Bījapallava* has long been recognized but hitherto it has only been studied in parts and that too to a very limited extent. Dr. Sita's study is the first to treat the work comprehensively in its entirety. It provides us with a clear image of Kṛṣṇa's algebra, dwelling on his merits as well as shortcomings. One of the remarkable facts revealed by her is that Kṛṣṇa did not know that one of Bhāskara's examples for the *varga-prakṛti* does not have integer solutions (see Example 2 in §4.9.2.2 of the present work).

A researcher of the history of mathematics is not a treasure hunter: he or she should be a geologist, capable of unravelling the different historical layers, rather than a gemologist. Dr. Sita is qualified to be a geologist in the discipline of the history of mathematics. I hope further serious studies on the *Bījapallava* in particular and on the Indian algebra in general by herself as well as by others will follow this valuable work of Dr. Sita.

January 1, 2010

Takao Hayashi
Professor of History of Science
Doshisha University
Kyoto, Japan.

APPRECIATION

The present book, *Bījapallava of Kṛṣṇa Daivajña*, is an outcome of the thesis by Dr. Sita for a Ph.D. degree of the University of Madras. *Bījapallava* itself is a critical commentary on the *Bījagaṇita* of Bhāskarācārya. Falling short of an exhaustive translation of *Bījapallava*, the present book is a comprehensive critical study of the work. Prof. Takao Hayashi, who has immensely contributed to the study of 'ancient' Indian mathematics starting with his scholarly and monumental work on the *Bakshali Manuscript*, has complemented the contents of the book with historical details of the time and place of the author Kṛṣṇa of the *Bījapallava* as also his other works. It was Dr. Sita's fortune that he was the examiner for the thesis and helped to improve it to a great extent. The historical details supplied by Prof. Hayashi make it clear that astronomers and mathematicians in the north of India right from the Vidarbha region took up the strings laid down by Āryabhaṭa, concentrating on the algebraic aspects of mathematics during the medieval period, while those in the south, particularly in the region now called Kerala, to name a few, Mādhava, Nīlakaṇṭa, Jyeṣṭadeva, Śaṅkaravarma, concentrated on finer aspects of the Āryabhaṭan tradition, sowing the seeds for analysis in the form of approximation. The latter also started the process of *Dṛg*, *Parahita* systems in astronomy having noted the need for corrections in the findings of earlier workers.

Dr. Sita, who was encouraged to take up a critical study of *Bījapallava* by me and had some initial hesitation about her background knowledge, has done full justice to the work and shown her mettle, as the contents of the book reveal.

Traditional Sanskrit studies limited to Tarka-Vyākaraṇa-Mīmāmsā, though essential for preserving the tradition, need to be augumented in the context of present day requirements by studies revealing the relevance of Sanskrit in the context of contributions in the language relating to exact and social sciences. The Kuppuswami Sastri Research Institute, a premier institution for Sanskrit research has realized this need and has been of late contributing significantly to this aspect.

The recent publication of *Sadratnamāla* of Śaṅkaravarman, by the Institute is a landmark towards this end. It is remarkable that Dr. Sita's book is published soon thereafter. Much of this activity, which I am sure would continue without barriers, owes its institution and culmination to the able leadership of Dr. Kameswari, the present Director of the Institute, backed by its management. Unlike earlier studies where some of the works in Sanskrit were taken up in pieces, I would plead that earlier work in the language relating to sciences are taken up as a whole, with complete translation and analytical study for publication. There is no dearth of such material craving for exposure to the world at large. Wishing the best to the Institute and its members in this venture,

22 February, 2012

M.S. Rangachari
Former Director & Head
The Ramanujan Institute for
Advanced study in Mathematics
University of Madras
Chennai

PREFACE

I take great pride in writing the preface to this publication of the Institute, '*Bījapallava of Kṛṣṇa Daivajña*', critically studied by Dr. Sita Sundar Ram. She was awarded the Ph.D. degree by the University of Madras for the same, prepared and submitted under my supervision, in the year 2008. The thesis was much appreciated by the examiners and Dr. T. Hayashi, one of the examiners (also has written the Foreword here) strongly recommended the publication of the thesis at the earliest. Accordingly, it was taken up for publication by the Institute.

The Institute has been publishing, from its inception, critical editions of texts related to Indian mathematics and astronomy. *The Journal of Oriental Research*, Madras, published by the Institute also contains articles pertaining to these subjects. Savants like Prof. T.S. Kuppanna Sastri, Prof. D.D. Kosambi, Dr. David Pingree, Dr. Arka Somayaji, Dr. K.V. Sarma and Dr. George Abraham, were involved in such academic endeavours of the Institute. Presently, Dr. M.S. Rangachari, Dr. S. Kannan, Dr. S. Madhavan, Dr. Takao Hayashi, Dr. M.D. Srinivas and others are also associated with the Institute.

The text *Sadratnamālā* of Śaṅkaravarman, on Indian astronomy and mathematics, with English translation and notes by Dr. S. Madhavan (Former Prof. of Mathematics, University College, Thiruvananthapuram), was published by the Institute, recently in 2011.

'*Bījapallava of Kṛṣṇa Daivajña*' is a landmark in this continued research activity on the part of the Institute in the field of Indian mathematics and astronomy.

It was possible for Dr. Sita to produce such a work, since she is a graduate in Mathematics and post-graduate in Sanskrit. The knowledge of both the subjects helped her a lot in understanding the mathematical texts in Sanskrit. Registered for the Ph.D. in 2001 at the Institute under my supervision, Dr. Sita studied the text more than once with me. In some places we met with the difficulty of presenting our understanding of the text in mathematical language.

Consultations with Dr. M.S. Rangachari (Former Director and Head, the Ramanujan Institute for Advanced Study in Mathematics, University of Madras), helped us a lot in synchronising the ancient mathematical methods with the modern ones. He always had the idea that Dr. Sita's thesis should definitely be published and accordingly suggested changes and modifications in the thesis to suit the publication. I am indeed grateful to him, both as Supervisor of Dr. Sita's thesis and as the Director of the Institute.

I am deeply beholden to Dr. Takao Hayashi (Professor of History of Science, Doshisha University, Kyoto, Japan), who had very willingly accepted to write the Foreword to the book on our request and also for his valuable suggestions.

The Institute is grateful to Mr. K.S. Sundar Ram for the financial support rendered by him towards the publication of the book.

Mr. B. Ganapathy Subramanian of the Madras Sanskrit College, Mrs. Srividhya of the Institute and Ms. K. Vidyuta (Post-graduate student in Sanskrit), are to be thanked for the layout and type-setting of the entire text. My thanks are also due to the Research scholars of the K.S.R.Institute for helping the editorial committee in various ways.

M/s. Sri Harish Printers are also to be thanked for neat printing and nice get-up of this book.

22.2.2012

Chennai - 4.

V. Kameswari

Director.

PREFACE TO THE SECOND REPRINT

The Institute takes great pleasure in bringing out this second reprint of the book, *Bījapallava of Kṛṣṇa Daivajña,* a representative work of the Algebra of 16th Century India. It was released during March 2012. It had been received well by the scholars in this special field of study all over the world and by the lovers of Sanskrit and Mathematics alike.

The Institute feels, that the efforts taken by Dr. Sita Sundar Ram in formulating the thesis which had been awarded the Degree of Ph.D. by the University of Madras, have been rewarded well. The book had seen three prints within a year's time.

Continuing this line of research activities, the Institute has initiated a group of serious scholars to work on various topics related to Mathematics as well as other Sciences. The Institute also hopes to bring out monographs on such fields of study with an aim to introduce to the discerning public, the wealth of knowledge that is hidden in the treasure house of Sanskrit works.

12.09.2012 **V.Kameswari**

Chennai - 4. **Director**

PREFACE TO THE FIRST REPRINT

It is with great pride that the Institute is reprinting this book within six months of its release. Highly gratifying is the response shown by the lovers of Sanskrit and Mathematics from all over the world. This gives us great encouragement and enthusiasm to work further in this specific field and bring more publications of this nature.

The Institute also hopes to expand its research activities in the allied scientific fields like Astronomy, Physics, Chemistry, Botany, Zoology, Medicine and so on, as found in our ancient Sanskrit texts.

16.06.2012
Chennai - 4.

V.Kameswari
Director

AUTHOR'S NOTE

The importance of mathematics as a basic science cannot be over emphasized. Ancient Indians learnt mathematics as part of their education. In the beginning, *gaṇita* was included in the Vedāṅgas and subsequently texts on astronomy. Bhāskara II of 12th Cent. A.D. was one of the first to write separate texts on arithmetic and algebra *viz.*, *Līlāvatī* and *Bījagaṇita.*

During the 16th Cent. A.D., the need was felt to write commentaries on the mathematical texts of Bhāskara II. We thus find commentaries on *Līlāvatī* and *Bījagaṇita.* Of the commentaries on *Bījagaṇita*, *Sūryaprakāśa* of Sūryadāsa (partially) and *Bījapallava* of Kṛṣṇa Daivajña have been published. Of these, *Bījapallava* holds a unique position as it is often cited by later Indian mathematicians and astronomers and also modern scholars, both Indian and foreign, working on Indian mathematics.

A special feature of Kṛṣṇa's *BP* is that Kṛṣṇa adds in it the most wanted *upapattis* (proofs / methods) to Bhāskara's terse *sūtras.* This feature of the text refutes to a large extent the popular Western criticism that Indian mathematics does not provide proofs.

Being a student of mathematics and Sanskrit, I was advised to take up a mathematical text for critical study. On perusal of mathematical literature available, it was found that no independent study has been done

on the commentary *Bījapallava,* though it has been often referred to by later mathematicians. Hence this text was taken up for a detailed study for my research work. What started as an adventure into the unkown territory of the ancient Indian mathematics, has been one of thrilling discovery.

The history of Indian mathematics can be divided for convenience into – before and after Bhāskara II. It is heartening to note that right from the Vedic times, *gaṇita* was given prime importance. It was Āryabhaṭa of the 5th Cent. A.D., who dealt with mathematics separately. From then on, there were many mathematicians. The credit goes to Bhāskara II, for writing a separate treatise on Algebra, in the 11th Cent. A.D. Four hundred years later, Kṛṣṇa Daivajña felt the neccesity to give a commentary on the *Bījagaṇita* of Bhāskara II.

Kṛṣṇa, in his text *Bījapallava*, has many innovations to his credit. He explains addition and subtraction of positive and negative numbers, by representing them on the (well-known) 'number line'.

The linear Diophantine equations have been dealt with in India since the times of Āryabhaṭa under the name *kuṭṭaka*. Kṛṣṇa gives proofs while elucidating Bhāskara's rules for the same.

The crowning glory of Indian Mathematics, the successful treatment of the eqaution $Nx^2 + 1 = y^2$ (*varga-prakṛti*) using the *carkravāla* method has been elaborately dealt with. While working on this section, I came across many articles and papers on the subject by authors both Indian and foreign. I was able to echo Andre Weil's words, when he says: "what would have

been Fermat's astonishment if some missionary, just back from India had told him that his problem had been successfully tackled there by native mathematicians almost six centuries earlier."

Any middle school child is familar with Śrīdhara's rule for solving the quadratic equation. Though his Algebra text is now lost, Bhāskara gives this rule and Kṛṣṇa, the *upapatti* (proof).

In the chapter on *ekavarṇasamīkaraṇa* (linear equations with one unknown), the verse: '*Ṣaḍaṣṭa śatakāḥ*' (6.1.2) came in for a lot of discussion by later authors. This vouches for the popularity of the text *Bījapallava*.

Bhāskara is one of the few who dealt with linear equation with many unknowns, equations with higher powers of unknowns and equations with products of unknowns. Kṛṣṇa deals with every one of them giving proofs.

Kṛṣṇa was a profound scholar in mathematics as well as in philosophy, prosody and *vyākaraṇa* as one could see from his work *Bījapallava*. Kṛṣṇa's research acumen can be understood from the fact that he has carefully read more than one manuscript of *Bījagaṇita* to arrive at the correct reading. He has also justified this by showing that the mathematical examples could be solved only using the corrected readings.

The present work in addition to the critical study of *Bījapallava,* also contains seven Appendices including the sūtras of *Bījagaṇita* (for easy reference) and a Glossary of Technical Terms.

In the presentation of the study, the following norms have been followed :

1. The examples selected for study are illustrative and not exhaustive. Problems have been so chosen as to highlight Kṛṣṇa's contributions.

2. Kṛṣṇa uses the word *upapatti* in the sense of both proof and method : the same is followed in the study.

3. Modern algebraic notation has been used. However, one example worked in the Indian method with the Devanāgari notation has been provided in Appendix IV.

4. A link to modern mathematics has been given wherever necessary.

While preparing the critical study, three different editions of the text, namely, *Bījapallavam*, ed. by T.V. Radhakrishnan, *Navāṅkura*, ed. by Apate Dattatreya and *Bījagaṇita* of Bhāskara II with *Bījāṅkura*, ed. by Viharilal Vasisht, were used. Also, two manuscripts of *Sūryaprakāśa* of Sūryadāsa and a study on the same by Pushpa Kumari Jain (till the *kuṭṭaka* section), were consulted. In this context it is imperative to point out that Ms. Pushpa Kumari's claim (Introduction, 16-7) that, "it is clearly possible that Kṛṣṇa has been influenced by *Sūryaprakāṣa*", seems to be not based on valid reasons.

I am deeply indebted to Prof. M.S. Rangachari (Former Director, Ramanujan Institute of Advanced Study in Mathematics, University of Madras) for recommending the text *Bījapallava* for critical study, providing me with valuable suggestions and constructive criticism at every stage of my research work and in the publication of the same.

I wish to record my thanks to Dr. E.R. Rama Bai (Former Prof. and Head, Dept. of Sanskrit, University of Madras) for suggesting the field of research.

I am deeply indebted to Dr. Takao Hayashi (Professor of History of Science, Doshisha University, Kyoto, Japan) for agreeing to write the Foreword to the book. He has been instrumental in motivating me to publish my thesis as a book.

I acknowledge with gratitude the valuable guidance extended to me by Dr. V. Kameswari, for her meticulous supervision my thesis and for accepting the same to be published by the Institute. In this connection I also thank the authorities of the Institute for their consent to publish the same.

I wish to record my thanks to Dr. K.S. Balasubramanian and Dr. T.V. Vasudeva, Deputy Directors of the K.S.R. Institute for their help in my study of the text and in the presentation of the thesis.

My thanks are due to Dr. M. Raghu (Librarian, The Madras Sanskrit College), a Jyotisha Siromani, who taught me the rudiments of Indian mathematics through the text *Līlāvatī.*

My thanks are also due to Prof. S. Kannan (Pro-Vice Chancellor, Central University, Hyderabad) for his initial guidance in interpreting the text. I also wish to thank Prof. M.D. Srinivas (Chairman, Centre for Policy Studies, Chennai), Dr. M.S. Sriram (Retd. Prof. and Head, Dept. of Theoretical Physics, University of Madras) and Dr. Kapil Paranjpe (Matscience, Chennai) for providing me with articles relevant to my subject. My good friends Mrs. Rajeswari Thyagarajan and Mr. S. Rajesh helped me

in solving few important problems. I also thank Dr. Amartya Kumar Dutta (Indian Statistical Institute, Kolkata), Prof. S. Arunasundaram (Prof. of Vyakarana, The Madras Sanskrit College) and Mr. K. Srikant (Research Scholar, K.S.R.Institute) and Mr. C. S. Puranik, Pune.

I record my sense of gratitude to the authorities of the Madras University for giving me permission to publish the work independently.

I thank the authorities of Tanjore Maharaja Serfoji's Saraswati Mahal Library, Tanjore and The Adyar Library and Research Centre, Chennai for allowing me to make use of their libraries.

I am very grateful to all the members of my family for their full support, throughout the period of my research. Special thanks are due to my husband, Mr. K.S. Sundar Ram, for his meticulous reading of the thesis and for his valuable suggestions.

This work is DEDICATED to my father, Sri B. Venkataramani, a brilliant student of mathematics.

Sita Sundar Ram

CONTENTS

APPENDICES

BĪJAPALLAVA OF KṚṢṆA DAIVAJÑA

A CRITICAL STUDY

SOME ILLUSTRIOUS INDIAN MATHEMATICIANS

(referred to in this book)

Period	Mathematician	Title
5th Cent. AD	Āryabhaṭa	*Āryabhaṭīya*
7th Cent. AD	Brahmagupta Bhāskara I Unknown Author	*Brāhma-sphuṭa-siddhānta* *Āryabhaṭīyabhāṣya* *Bakshali Manuscript*
8th Cent. AD	Śrīdharācārya Padmanābha	*Bīja, Pāṭīgaṇita* *Bīja*
9th Cent. AD	Mahāvīra Pṛthūdakasvāmin	*Gaṇita-sāra-saṅgraha* *Bhāṣya on Brāhma-sphuṭa-siddhānta*
10th -15th Cent.	Āryabhaṭa II	*Mahāsiddhāntikā*
11th Cent. AD	Śrīpati Bhāskarācārya (Bhāskara II) Nārāyaṇa	*Gaṇitatilaka, Siddhāntaśekhara, Bījagaṇita* *Bījagaṇita, Līlāvatī, Siddhānta Śiromaṇi* *Nārāyaṇa Bīja*
14th Cent. AD	Nārāyaṇa Paṇdita Mādhava	*Gaṇita Kaumudī, Bījāvatamasa* *Veṇvāroha*
15th Cent. AD	Nīlakaṇṭa Somayāji	*Tantrasaṅgraha*
16th Cent. AD	Sūryadāsa KṚṢṆA	*Sūryaprakāśa* ***BĪJAPALLAVA***
17th Cent. AD	Jñānarāja Munīsvara Raṅganātha Kamalākara	*Siddhānta Sundarabīja* *Marīci, Pāṭīśāra* *Guḍhārthaprakāśaka* *Siddhāntatattvaviveka*
20th Cent. AD	Srinivasa Ramanujan	*Theory of Numbers.*

CHAPTER - 1
A BRIEF SURVEY OF INDIAN MATHEMATICS

यथा शिखा मयूराणां नागानां मणयो यथा ।
तथा वेदाङ्गशास्त्राणां गणितं मूर्धनि स्थितम् ।।

"Like the crest of the peacocks, the gems on the hoods of cobras, mathematics is at the top of the Vedāṅga Śāstras" – *Vedāṅga Jyotiṣa* (v.4).

The above statement makes it clear as to how Hindu society considered mathematics as an integral part of human life. A survey of the development of mathematical ideas from the ancient through medieval times upto the tradition in modern times, is presented here.

1.1. History of Indian Mathematics :

Mathematics, as everyone is aware, has a long tradition in ancient and medieval India. The *Ṛg Veda*, the earliest known text mentions numbers in many places. From the time man learnt to count, he has used arithmetic. In our own land, mathematics was used for very practical purposes. Religion and rites were predominant in Vedic society. The religious practices in ancient India required a certain amount of knowledge of astronomy and mathematics for astronomical studies and other vocations. For instance, the Vedic man used mathematics to set the auspicious time and so on; the *Śulba sūtras* dealt with mathematics to construct the *vedis* (altars) accurately. As it always happens, some of the scholars got interested in mathematics for its own sake. They took pains to study some of the aspects of the subject thoroughly and wrote commentaries or independent treatises.

The term '*gaṇita*' means the 'science of calculation'. The term is very ancient and one finds it in Vedic literature.[1]

1. B. Datta and A.N. Singh, *History of Hindu Mathematics*, Vol. II, Bharatiya Kala Prakashan, Delhi, 2001, p.2

One of the branches of Jaina religious literature is called *Gaṇitānuyoga*[2]. Knowledge of *saṅkhyāna* (science of numbers) was considered to be one of the principal achievements of Jaina priests. In Buddhist literature[3] also, arithmetic (*gaṇa saṅkhyāna*) was regarded as the first and foremost knowledge to be gained. Three classes of *gaṇita* are mentioned in Buddhist literature : 1) *mudrā* (finger arithmetic), 2) *gaṇanā* (mental arithmetic) and 3) *saṅkhyāna* (higher arithmetic).

In the initial stage of education in Hindu society, the main subjects of study were *lipi* or *lekhā* (alphabets, writing), *rūpa* (geometry) and *gaṇana* (arithmetic)[4]. The Haṭigumpha inscriptions mention that King Kharavela (163 B.C.) of Kaliṅga spent nine years in mastering the above[5]. It is also said in Kauṭilya's *Arthaśāstra* that after the ceremony of tonsure, the student shall learn *lipi* (alphabets) and *saṅkhyā* (arithmetic).[6]

The first significant mathematical text to be found in the Vedic literature, is the *Śulba sūtras*. In these are given rules with mathematical details for constructing the *vedis* or sacrificial altars. This text gradually introduces us to *karaṇīs* (surds) of the type $\sqrt{2}$, $\sqrt{3}$ etc. Both the *Baudhāyana* (I. 61-2) and *Āpastamba Śulba sūtras* (I.6) give very good rational approximation for the irrational $\sqrt{2}$ in the form $1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{1.3.4.34}$ correct to 5 decimal places.

Āryabhaṭa (5th Cent. A.D.) was the author of *Āryabhāṭīya*. He was the first to deal with indeterminate equations of first degree or *Kuṭṭaka*. He has also written a few important sections on mathematics in his work. "By doing so", says Venugopal D. Heroor, "the author started a tradition of

2. *ibid.*, p. 4
3. *loc.cit,*
4. *ibid.*, p. 6
5. *loc.cit.*
6. *Arthaśāstra* I.5.2 : वृत्तचौलकर्मा लिपिं सङ्ख्यानं चोपयुञ्जीत ।

including in astronomical treatises a separate and full chapter exclusively devoted to mathematics''[7]

The credit goes to Āryabhaṭa for giving for the first time, the value of π (*pi*) correct to four decimal places as equal to 3.1416. It is also remarkable that Āryabhaṭa uses the word *āsanna*, meaning approximate which definitely suggests that he knew that the value given by him was only approximate. This achievement of Āryabhaṭa in the 5th Century was truly significant in the light of the fact that it is only thirteen centuries later in 1761, that Lambert proved that π (*pi*) is irrational and only in 1882, Lindemann established that it is transcendental.[8]

Mahāvīra of the ninth century was a mathematician of high calibre. He was the first to give an independent status to mathematics by not treating it as a 'handmaid of astronomy.' He declared (*Gaṇita-sāra-saṅgraha*, Samjñādhikāra, v.9) :

लौकिके वैदिके वापि तथा सामायिकेऽपि यः ।
व्यापारस्तत्र सर्वत्र संख्यानमुपयुज्यते ।।

''In all those transactions which relate to Vedic or (other) similarly religious affairs, calculation is of use''[9].

Mahāvīra was the first to give the general formula for $^{n}c_{r} = \dfrac{n(n-1)\ ..\ (n-r+1)}{1.2.3.....r}$, the number of ways of choosing r things out of n things. He was also probably the first in ancient India to have discussed the ellipse. He called it *āyata vṛtta*.

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7. Venugopal D. Heroor, *The History of Mathematics and Mathematicians of India*, Vidya Bharati Karnataka, Bangalore, 2006, p. 65.

8. S. Balachandra Rao, *Indian Mathematics and Astronomy*, Jnana Deep Publications, Bangalore, 1994, pp. 39-40,

9. *Gaṇita-sāra saṅgraha* of Mahāvīrācārya, with Eng. Tr. and notes by M.Rangacharya, Govt. of Madras, 1912, p.2.

The *Bakṣālī Manuscript* (*BM*) is the name given to a mathematical treatise which was found near the village Bakṣālī. It is devoted mostly to arithmetic and algebra with just a few problems in geometry. One of the most important mathematical contributions found in *BM* is a formula for extracting the square-root of a non-square number. Though the author of the work is not known, its algebraic contents roused the interest of mathematicians all over the world.[10]

Brahmagupta's (7th Cent. A.D.) work is called *Brāhma-sphuṭa-siddhānta* (*Br.Sp.*). Brahmagupta dealt with zero and negative numbers for the first time in ancient Indian mathematics. He was also the first to give a partial solution to the problem of solving $Nx^2 + 1 = y^2$ through his method of *bhāvanā*. Admiring the process of *bhāvanā*, L.E. Dickson records in the preface to his book on *History of the Theory of Numbers* : "It is a remarkable fact that the Hindu Brahmagupta in the seventh century gave a tentative method of solving $ax^2 + c = y^2$ in integers which is a far more difficult problem than its solution in rational numbers."[11]

Śrīpati (11th Cent. A.D.) is the author of three works, *Gaṇita Tilaka, Siddhānta śekhara* and *Bījagaṇita. Gaṇitatilaka* is a treatise on arithmetic and *Bījagaṇita* is now lost. Two chapters of *Siddhānta śekhara* are devoted to arithmetic and algebra. The fourteenth chapter on algebra is especially important because Śrīpati introduces many innovations in *kuṭṭaka* method in indeterminate equations of first degree and *varga-prakṛti* (indeterminate equations of second degree).

10. Details on this text can be had from Takao Hayashi's recent edition, *The Bakhshālī Manuscript* (An ancient Indian mathematical treatise) published by Egbert Forsten, Groningen, The Netherlands, 1995.

11. Quoted by Amartya Kumar Dutta, "Brahmagupta's Lemma: The Samasabhavana," *Resonance* (Nov. 2003), pp. 10-24.

Two more works, the *Bīja* of Śrīdharācārya and that of Padmanābha, though quoted by Bhāskarācārya are now lost to us.

With the advent of Bhāskarācārya (Bhāskara II) (hereafter Bhāskara) in the 12th Century, Indian mathematics reached the pinnacle of its achievement. This was the discovery of the method for solving the (erroneously called) Pell's equation. $Nx^2 \pm 1 = y^2$, where N is a non-square positive integer and x and y are also required to be integers. This was known as the *cakravāla* method or cyclic method. Bhāskara himself in his *Bījagaṇita* (*BG*) declares *cakravālamidam jaguḥ* – meaning, it was known to people before him. Quite recently it has been proved that it was known to a scholar by name Jayadeva before Bhāskara. However, it was Bhāskara who dealt with it elaborately. To use Brahmagupta's method for the equation $Nx^2 \pm k = y^2$, there had to be an initial auxiliary equation which Brahmagupta could only find by trial and error method. Bhāskara achieved remarkable success when he evolved a simple and elegant method which helped to derive an auxiliary equation having the required interpolators k as ± 1, ± 2, ± 4, simultaneously with two integral roots from another auxiliary equation empirically formed with any simple value of the interpolator, positive or negative. This method was called *cakravāla*, so called because "it proceeds as in a circle, the same set of operations being applied again and again in a continous method".[12]

C.O. Selenius, a modern mathematician, gives an appreciative analysis of this method thus: "The old Indian *chakravāla* method for solving the mathematically fundamental indeterminate *varga-prakṛti* equation was a very natural, effective and labour-saving method with deep-seated mathematical properties.

"The method represents a best approximation algorithm of minimal length that, owing to several minimization properties, with minimal effort

12. Datta and Singh, *op.cit.*, p.162

and avoiding large numbers always automatically (without trial processes) produces the least solutions to the equation, and thereby the whole set of solutions . . .

"It is accepted that the *chakravāla* method here explained, anticipated the European methods by more than a thousand years. But, as we have seen, no European performances in the whole field of algebra at a time much later than Bhāskara's, nay nearly up to our times, equalled the marvellous complexity and ingenuity of *chakravāla.*"[13]

Bhāskara was also the first Indian astronomer-mathematician to introduce into an astronomical text, the moon's equation which is now called evection. "It is remarkable that Bhāskara's discovery preceded that in the west (by Tycho Brahe) by nearly four centuries", adds S. Balachandra Rao.[14]

Around 12th Century, the Indian mathematicians concentrated on producing *Siddhāntas* or treatises on astronomy. Separate texts of mathematics were few and far between. In addition to *Nārāyaṇa Bīja* of Nārāyaṇa, *Siddhānta Sundarabīja* of Jñānarāja and *Gaṇita Kaumudī* and *Bījāvatāmsa* of Nārāyaṇa Paṇḍita, mention should be made here, of an unbroken tradition in Indian mathematics after these authors. More sophisticated ideas of analysis such as approximations (inclusive of one to π) were treated by mathematician - astronomers like Mādhava (14th Cent. A.D.), Nīlakaṇṭha (15th Cent. A.D.) etc., atleast two to three centuries earlier to Newton and others.

No history of Indian Mathematics can be complete without mentioning the great mathematician of the 20th Century, Srinivasa Ramanujan. He did great work on the theory of numbers inclusive of

13. Clas Olaf Selenius, "The Rationale of the Chakravāla process of Jayadeva and Bhāskara II", *Historia Mathematica 2* (1975), pp. 167-84.

14. *ibid.*, p. 141.

partitions and related themes. He was "discovered" by the English mathematician G.H. Hardy and invited to England where he did most of his work. Like his ancestors, Ramanujan hardly wrote proofs for his findings. Scholars all over the world are still working on his notebooks.

Starting from *Śulba sūtras*, Āryabhaṭa, Mahāvīra, Brahmagupta, Śrīpati, Śrīdhara and Padmanābha have contributed directly and indirectly to the growth of algebra. Of these scholars, Bhāskara stands out for having treated algebra as a separate genre of mathematics.

Also, it was Bhāskara who for the first time, classified *bījagaṇita* into two as : tools of algebra and its application. At the end of the first section of *BG* (v.101)[15] he says:

उक्तं बीजोपयोगीदं संक्षिप्तं गणितं किल ।
अतो बीजं प्रवक्ष्यामि गणकानन्दकारकम् ।।

"The section of this science of calculation which is essential for analysis has been briefly set forth. Next I shall propound analysis, which is the source of pleasure to the mathematician"[16]

1.2. Distinction between Arithmetic and Algebra :

The distinction between arithmetic and algebra, to some extent can be found in their special names. Arithmetic deals with mathematical operations while algebra deals with determinations of unknown entities. Hence arithmetic was known as *vyaktagaṇita* and algebra as *avyaktagaṇita.*

1.3. Treatises on Arithmetic :

Earlier, arithmetic was known by the name *pāṭīgaṇita,* the science of calculation which required a board or *pāṭī.* The works currently available

15. References to the verses of *BG* of Bhāskara are to the Text (without Bhāskara's gloss) given in Appendix VI.
16. Datta and Singh, *op.cit.*, p.6.

which deal with arithmetic are : *BM* (200 - 700 A.D.)[17] the *Triśatikā* of Śrīdharācārya (8th Cent. A.D.), the *Gaṇita-sāra-saṅgraha* (*GSS*) of Mahāvīra (9th Cent. A.D.), the *Gaṇitatilaka* of Śrīpati (11th Cent. A.D.), the *Līlāvatī* of Bhāskarā, the *Gaṇita Kaumudī* of Nārāyaṇa Paṇḍita (14th Cent. A.D.) and the *Pāṭīsāra* of Munīśvara (17th Cent. A.D.).[18]

1.4. Treatises on Algebra :

The early Indians regarded algebra as a science of great importance and utility. Calling algebra by the name '*bījagaṇita*', Bhāskara says : '*ekameva matirbījam*' "intelligence alone is algebra". He adds in his gloss (*BG.* p. 100)[19] :

बीजं मतिरिति । हि यस्मात् कारणात् बुद्धिरेव पारमार्थिकं बीजं वर्णास्तु तत्सहायाः । गणककमलतिग्मरश्मिभिराद्यैराचार्यैर्मन्दावबोधार्थमात्मीया या मतिर्विविधवर्णान् सहायान् कृत्वा विस्तारं नीता सैवेह संप्रति बीजगणितसंज्ञां गता ।

i.e. intelligence is certainly real analysis ; symbols are its helps. "The innate intelligence which has been expressed for the duller intellects by the ancient sages, who enlighten mathematicians as the sun irradiates the lotus, with the help of various symbols, has now obtained the name of algebra".[20]

He also remarks (*BG.* p. 127) :

उपपत्तियुतं बीजगणितं गणकाः जगुः ।
न चेदेवं विशेषोऽस्ति न पाटीबीजयोर्यतः ।।

i.e., mathematicians have declared algebra to be calculation accompanied by proofs; otherwise, there would be no distinction between arithmetic and algebra.

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17. Refer T. Hayashi's edition of *BM*, pp.148-49.
18. Datta and Singh, *op.cit.*, Vol. I. p.125.
19. References to the gloss of *BG* by Bhāskara are to the edition of *BG* by Sudhākara Dvivedi with expository notes and illustrative examples, and with further notes by Muralidhara Jha, Benares Sanskrit Series No. 159, Benares, 1927.
20. Datta and Singh, *op.cit.*, Vol. II. p.2.

In Bhāskara's *BG,* the initial chapters deal with *dhanarṇaṣaḍvidha* (law of signs), *khaṣaḍvidha* (laws of zero and infinity), *avyaktaṣaḍvidha* (operations of unknowns) *karaṇī* (surds), *kuṭṭaka* (indeterminate equations of first degree), *varga-prakṛti* and *cakravāla* (indeterminate equations of second degree). In the latter section, Bhāskara teaches the application of the *sūtras* stated earlier.

1.5. Commentaries :

In olden days, the young student had to do *gurukulavāsa* in order to study. A person interested in *gaṇita* was first made to commit all the rules to memory. He probably did all the calculations on a *pāṭī*. This way, he mastered the rudiments of mathematics, having been guided by his master. A certain amount of elementary mathematics was learnt by all. It was necessary for all the religious ceremonies. There were, of course, some who were interested in the subject for its sake and made a thorough study of it.

Often the *mūlagrantha* contained only a general enunciation ; the pupils learnt the proofs and applications sitting in front of the teacher. Also, for the sake of brevity and for retention in memory, often the *sūtras* were cryptic.

As is customary in Indian tradition, commentaries grew up along with original texts in the field of mathematics too. Starting from Bhāskara I who wrote the commentary on *Āryabhāṭīya*, the commentaries made their own contribution to the development of Indian mathematics.

It is interesting to note that the *BG* of Bhāskara has attracted the attention of many a scholar, more than any other work. Many commentaries were written on it. The following list[21] gives the details :

21. S.N. Sen, *A Bibliography on Sanskrit Works on Astronomy and Mathematics,* National Institute of Sciences of India, New Delhi, 1966.

1. *Bālabodhinī* or *Bījodāharaṇa* or *Bījagaṇitavivṛti* or *Bījagaṇitavyākhyā* – by Kripārāma Miśra, s.o. Lakṣminārāyaṇa and a resident of Ahmedabad (c. 1792 A.D.)
2. *Bījapallava* or *Bījavivṛtikalpalatāvatāra* or *Kalpalatāvatāra* or *Bījanavāṅkūra* – by Kṛṣṇa Daivajña, s.o. Ballāla (Early 17th Cent. A.D.).
3. *Bījavivṛtikalpalatā* – by Paramasukha, s.o. Sītārāma (Early 19th Cent. A.D.)
4. *Bījaprabodha* – by Rāmakṛṣṇa, s.o. Lakṣmaṇa, grandson of Nṛsimha and pupil of Somanātha.
5. *Bījagaṇita-ṭīkā* – by Sūrya Daivajña
6. *Sūryaprakāśa-bīja vyākhyā* (1541 A.D.) – by Sūryadāsa also known as Sūrya, s.o. Jñānarāja (late 16th Cent. A.D.).

Of these *Bījapallavam (BP)* of Kṛṣṇa Daivajña (hereafter Kṛṣṇa) seems to have been the most popular, since it has been referred to by *gaṇitajñas* after him and also by both Indian and foreign scholars of the modern times.

1.6. Kṛṣṇa Daivajña, his lineage and his Works :

Kṛṣṇa who came five centuries after Bhāskara, felt the need for an elaborate commentary on *BG*. He very aptly called his work *Bījapallava* or *Bījāṅkura*. He obviously felt, that the tree of algebra was not yet dead; but he had to keep it alive so that it came forth with sprouts and leaves[22].

1.6.1. Kṛṣṇa's account of himself in his *granthasamāpti* :

अभूत्पृथिव्यां प्रथितो गुणौघैश्चिन्तामणिर्दैवविदां वरिष्ठः ।
संपूजनानेहसियस्य गौरी स्मृता स्तुता प्रत्यहमाविरासीत् ।।

22. T.V. Radhakrishna Sastri, ed. *Bījapallavam*, Tanjore Saraswati Mahal Series, No. 78, Tanjore, 1958. Preface p.iii.

तत्सूनवः पंच बभूवुरेषां ज्येष्ठोभिरामः किल रामनामा ।
भविष्यदर्थज्ञतया हि यस्य विदर्पराजोपि निदेशवर्ती ।।

रामादभूतां सीतायां पुत्रौ कुशलवाविव ।
त्रिमल्लो गोपिराजश्च गुणैः सर्वैः समन्वितौ ।।

त्रिमल्लसूनुर्जयति द्विजेन्द्रो बल्लालसंज्ञः शितिकण्ठभक्तः ।
यः सन्ततं रुद्रजपाति संङ्गाद् ब्राह्मं महोमूर्तमिवावभाति ।।

दैवज्ञवर्यगणसन्ततसेव्यपार्श्व बल्लालसंज्ञगणकस्य सुतोऽस्ति कृष्णः ।
रामानुजः स परमेश्वरतुष्टिहेतोर्बीजक्रिया विवृतिकल्पलतामकार्षीत् ।।

He was the son of Ballāla a great devotee of Lord Śaṅkara who was always doing *rudrajapa* and was himself a mathematician. Kṛṣṇa's elder brother was Rāma. His grandfather was Trimalla. Kṛṣṇa says he wrote the text *Bījapallavam* in order to please Lord Śiva.

1.6.2. On his preceptors (vv. 6-10) :

आसीदसीमगुणरत्ननिधानकुम्भः कुम्भोद्भवाभरणदिग्ललनाललाम ।
आशैशवार्धितविशेषकलानुवर्ती श्रीकेशवः सुगणितागमचक्रवर्ती ।।

तस्मादभूद्भुवनभूषणभूतमूर्तिः श्रीमानगण्यगुणगौरवगेयकीर्तिः ।
ज्योतिर्विदागमगुरुर्गुरुसंप्रदायप्रज्ञानशास्त्रहृदयः सदयो गणेशः ।।

भ्रातुः सुतस्तस्य यथार्थनामा नृसिंह इत्यद्भुतरूपशोभः ।
अवर्धयद्यो जगतामभीष्टं प्रह्लादमाश्चर्यकरः सुराणाम् ।।

तच्छिष्यो विष्णुनामा स जयति जगतीजागरूकप्रतिष्टः
शिष्टानामग्रगण्यः सुभणितगणिताम्नायविद्याशरण्यः ।
यद्वक्त्रोन्मुक्तमुक्ताफलविमलवचोवीचिमालागलन्तो -
र्द्वित्राः सिद्धान्तलेशा जगति विदधतेऽज्ञेऽपि सर्वज्ञगर्वम् ।।

तस्मादधीत्य विधिवत् त्रिस्कन्धं ज्योतिषं गुरोः ।
कृष्णो वेदविदां श्रेष्टस्तनुते बीजपल्लवम् ।।

अव्यक्तत्वादिदं बीजमित्युक्तं शास्त्रकर्तृभिः ।
तद्व्यक्तीकरणं शक्यं न विना गुर्वनुग्रहम् ।।

He first pays obeisance to Varāhamihira and Bhāskarācārya. There was once a Śrī Keśavā who was a king of Mathematics – '*sugaṇitāgama-cakravartī*'. His son was Gaṇeśa, a *gaṇaka* par excellence. Gaṇeśa's brother's son was called Nṛsimha. Nṛsimha's pupil was Viṣṇu Daivajña who had mastered mathematics and all the *siddhāntas*. Kṛṣṇa records that he learnt the *triskandha jyotiṣa* (*Siddhānta, Samhitā* and *Horā*) from him and wrote the *Bījapallavam*. This treatise on algebra, Kṛṣṇa admits could not be possible without the grace of his revered *guru*.

1.6.3. Date of Kṛṣṇa :

1) Kṛṣṇa lived in the late 16th and early 17th Century. According to *New Catalogus Catalogorum* (*NCC*)[23], Vol. IV. (p.323) the following facts about Kṛṣṇa are known :

"Son of Ballālagaṇaka ; brother of Rāma, Govinda, Raṅganātha and Mahādeva; uncle of Munīśvara alias Viśvarūpa, son of Raṅganātha, grandson of Trimalla and disciple of Viṣṇu ; patronised by Emperor Jehangir''.

2) While enumerating the different works by Kṛṣṇa, the above source states that he wrote a commentary *Udāharaṇa* on the *Jātaka-paddhati* of Śrīpati, mentioning the name of Khan Khan and the date 1556 A.D. clearly.

3) This fact is confirmed by Kṛṣṇa's brother Raṅganātha at the end of his work *Gūḍhārthaprakāśaka* on *Sūryasiddhānta*[24] :

ततः स कृष्णो जहँगीरसार्वभौमस्य सर्वाधिगतप्रतिष्ठः ।
श्रीभास्करीयं विवृतं तु येन बीजं तथा श्रीपतिपद्धतिः सा ।।

23. *N.C.C.* Vol. IV. University of Madras, 1968.

24. *Sūryasiddhānta* with the commentary *Gūḍhārthaprakāśaka* of Raṅganātha, Calcutta, 1871, p. 187.

He also says that their father Ballāla lived on the banks of the river Bhāgīrathī in Benares :

भागिरथीतीरसंस्थे शम्भोर्वाराणसीपुरे ।
बल्लालगणको रुद्रजपासक्तोऽभवद्बुधः ॥

4) R.C. Majumdar corroborates this view and confirms that Kṛṣṇa was the chief astrologer in King Jahangir's court.[25]

5) Munīśvara, son of Raṅganātha referred above, gives more details about his family in his commentary *Marīcī* on the *Golādhyāya* of Bhāskara. He states that his forefather Cintāmaṇi Daivajña belonged to the Devarātragotra of Dadhigrāma on the Payoṣṇī river[26].

1.6.4. Works of Kṛṣṇa :

NCC (Vol. IV, p. 323) also lists the different works written by Kṛṣṇa

1. (*Bījavivṛtti*) *Kalpalatāvatāra,* (*Bīja*)*pallava* or (*Bīja*)*aṅkura* or *Navāṅkura* – commentary on *Bījagaṇita* of Bhāskarācārya.
2. *Chādakanirṇaya* – on eclipses.

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25. R.C. Majumdar, *History of Culture of the Indian People,* Volume VII, *The Mughal Empire* (III Edition), 1994, Chapter I, p. 15.

26. *Golādhyāya* of Bhāskara with the comm. *Marīcī* by Munīśvara, Anandasrama Series 122, Pt. II, 1952, p.527 (vv. 1-2, 8) :

गङ्गाभौशिलनगराद्गोदावर्यनुगतादुपायातः ।
एलचपुरसमदेशे तटे पयोष्ण्याः शुभे दधिग्रामे ॥

ज्योतिर्विद्याविद्भि सकलैः शिरसा सदा बुधैर्मान्यः ।
व्यजयत यजुः श्रुतिज्ञो गणकश्चिन्तामणिर्नाम्ना ॥

तस्यानुजः सकलशास्त्रसरोजभृङ्गो बीजक्रियाविवृतिकल्पलतानिदानम् ।
श्रीनूरदीनपरमप्रणयैकपात्रं कृष्णो बभूव जनिपद्धति वृत्तिकारः ॥

3 *Janipaddhativṛtti* – referred to along with *Chādakanirṇaya* by Munīśvara in his commentary *Marīcī* on *Golādhyāya.*

4. *Udāharaṇa* – a commentary on the *Jātakapaddhati* of Śrīpati.

1.7. Contents of *Bījapallavam* :

As the title suggests (*bīja* meaning algebra) the text *Bījapallavam* (*BP*) is devoted wholly to algebra. It is an excellent commentary on the *BG* of Bhāskara. Kṛṣṇa quotes the *sūtras* and examples verbatim and then explains them in lucid prose.

A profound scholar that he is, Kṛṣṇa's style is simple though technical. He shows great respect to his Ācārya. His detailed commentary is faithful to the original though he has contributed new methods especially in the chapters on *Kuṭṭaka* and *Cakravāla*. In the chapter on '*Anekavarṇasamīkaraṇa*' he has added two examples of his own.

BP comprising of thirteen chapters is broadly divided into two sections.

a) the first seven chapters teach the tools of algebra :

1) Six fundamental operations (*ṣaḍvidha*) of positive and negative numbers.

2) *Ṣaḍvidha* of zero, and infinity.

3) *Ṣaḍvidha* of unknown quantities. The unknown quantities are represented by symbols, which are letters of the Sanskrit alphabets.

4) The various definitions for the word *karaṇī* and its operations.

5) Devoted wholly to solving indeterminate equations of first degree (*kuṭṭaka*) and its various forms.

6) The *varga-prakṛti* or indeterminate equations of second degree.

7) A special case of *varga prakṛti* namely *cakravāla* is taken for discussion. This was the major contribution of Bhāskara elucidated by Kṛṣṇa with his original ideas.

b) The subsequent chapters explain the rationale or applications of the formulae propounded in the earlier chapters :

8) Linear equations with one variable – *ekavarṇasamīkaraṇam*

9) Equations of higher degree with one variable – *madhya-māharaṇa.*

10, 11) Explain equations with several unknowns – *anekavarṇa-samīkaraṇa* – the tenth to linear equations and eleventh to equations of higher powers of unknowns.

12) Equations involving products of unknowns are taken up for study. These are called *bhāvita.*

13) *Granthasamāpti* – is devoted to the conclusion of the text by Bhāskara about his text and commentator Kṛṣṇa's details about himself.

1.8. Editions of *BP* used for Study :

1) *Bījapallavam,* Ed. T.V. Radhakrishnan, Tanjore Saraswathi Mahal Series No. 78, Tanjore, 1958.

2) *Navāṅkura,* by Apate Dattatreya, Anandasrama Sanskrit Series 99, Pune, 1930.

3) *Bījagaṇita* of Bhāskara II with *Bījāṅkura,* ed. by Viharilal Vasisht, Ranbir Kendriya Samskrta Vidyapeetam, Jammu, 1982.

1.9. Editions of *BG* referred :

While comparing the many editions of *BG,* it is found that Bhāskara's own gloss has been included in the following editions. In fact, Mm. Sudhakara Dvivedi has very clearly segregated Bhāskara's gloss from his own *ṭīkā.* A study of Bhāskara's gloss becomes imperative in view of the fact that often Kṛṣṇa directs the reader to refer *BG* for clarification.

1) *Bījagaṇita* of Śrī Bhāskarāchārya (with his own gloss) and with expository notes and illustrative examples by Mm. Pandit Sudhakara Dvivedi, edited with further notes by Mm. Muralidhara Jha, Benares Sanskrit Series No. 159, Benares, 1927.

2) ————, ed. Jivananda Vidyasagara, Calcutta, 1878.[27]

The third text given below in addition to Bhāskara's gloss copiously quotes from Kṛṣṇa's *BP* for clarifying many issues :

3) ————, ed. Achyutananda Jha, with Jivanatha Jha's *Subodhinī* Sanskrit Com. and with Acyutananda Jha's *Vimalā,* Sanskrit and Hindi Com. Kasi Sanskrit Series 148, Chowkamba Sanskrit Samsthan, Varanasi, 2002.

4) *Bījagaṇita,* text with English tr. by S.K. Abhayankar, Bhaskariya Prathistana, Pune, 1980.

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27. This edition of *BG* with Bhāskara's own gloss, as edited by Jivananda Vidyasagara, has some portions of the com. of Kṛṣṇa in it. Dr. Pushpa Kumari Jain, however in her edition of *The Sūryaprakāśa* of Sūryadāsa takes this edition as the correct text of *BG.* There is a discussion on this in Appendix I.

So far, a concise history of mathematics, specifically algebra, right from the time of the Vedas upto Srinivasa Ramanujan of 20th century has been given. Also, the important mathematicians and their works and the mathematical developments through the ages have been brought out. Bhāskara's achievements in the field of algebra, the texts written exclusively on *Pāṭīgaṇita* and *Bījagaṇita* and the important commentaries on *Bījagaṇita* have been mentioned. The life and contributions of Kṛṣṇa to the field of mathematics and a brief account of the contents of the text, *Bījapallava* are provided.

CHAPTER - 2
SIX MATHEMATICAL OPERATIONS

Kṛṣṇa had introduced the number line to elucidate the addition and subtraction operations on positive and negative integers, much ahead of such an idea in Western mathematics. The concepts of zero and infinity as explained by Kṛṣṇa and use of symbols in the form of letters of the alphabet for 'unknown' or 'entities to be solved' as suggested by Kṛṣṇa are some of his worthy contributions. The mathematical cperations on surds as given by Kṛṣṇa shows that the modern way of rationalization was known to Bhāskara and Kṛṣṇa and perhaps to even earlier writers.

That all mathematical operations are variations of the two fundamental operations of addition and subtraction was recognised by the Indian mathematicians from early times. Bhāskara I states[1] "so previous teachers have said : Indeed every mathematical operation will be recognised to consist of increase and decrease. Hence the whole of this science should be known as consisting truly of these two only"[2]. It is to be noted that 'increase' means 'addition'; and 'decrease' 'subtraction' in present day language.

The number of fundamental operations in algebra is recognised by Indian algebraists to be six *viz.*, addition, subtraction, multiplication, division, squaring and extraction of the square root. Thus in *Pāṭīgaṇita* or arithmetic, cubing and extraction of cube root are included making the total eight. The last two are excluded in algebra. However almost all

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1. *Āryabhaṭīya* with the comm. of Bhāskara I and Someśvara ed. with Introduction and Appendices by K.S. Shukla, Indian National Science Academy, New Delhi, 1976, p. 43 :

   संयोगभेदा गुणनागतानि शुद्धेश्च भागो गतमूलमुक्तम् ।
   व्याप्तं समीक्ष्योपचयक्षयाभ्यां विद्यादिदं द्व्यात्मकमेव शास्त्रम् ॥

2. Datta and Singh, *op.cit.*, Vol. I. p.130.

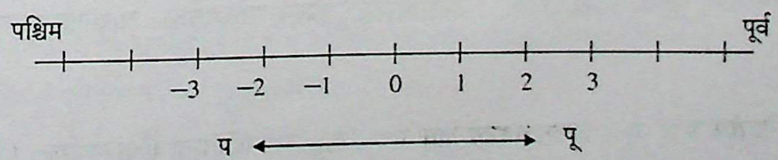
Indian authors on arithmetic have the formula for $(a+b)^3$. In fact Mahāvīra even extends the formula to more than two terms.

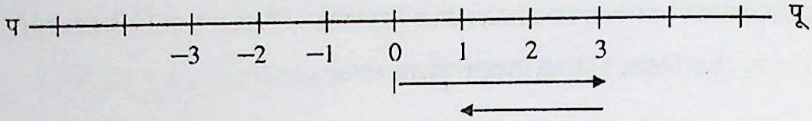
2.1. *Dhanarṇaṣaḍvidha* (Law of signs) :

1) This section explains the six fundamental operations dealing with positive and negative numbers. While *dhana* (wealth) and *sva* meant positive numbers, *ṛṇa* (debt) and *asva* denoted negative numbers.

2) Brahmagupta was the first to give in detail the operations involving positive and negative numbers. The *saṅkalana* rule is quite straightforward. Kṛṣṇa explains the *vyavakalana* quite elaborately (*BP.* p. 13) : ऋणत्वमिह त्रिधा तावदस्ति देशत: कालत: वस्तुतश्चेति . . . तच्च वैपरीत्यमेव । ... तत्रैकरेखा स्थिता द्वितीया दिक् विपरीता दिगित्युच्यते । यथा पूर्वविपरीता पश्चिमा दिक् । यथा उत्तरदिग्विपरीता दक्षिणा दिगित्यादि । तथा च पूर्वापरदेशयोर्मध्ये एकतरस्य धनत्वे कल्पितं तं प्रति तदितरस्य ऋणत्वम् ।। – Negation is of three sorts, according to place, time and things. It is in short contrasting; just as west is the contrary direction to east and south to north. Thus of two places situated in the east and west, if one is taken to be positive, the other is relatively negative.

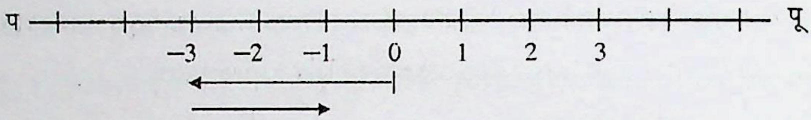
Thus Kṛṣṇa explains addition and subtraction of negative and positive numbers with the help of identifying them with points on a straight line. Modern school texts call this "number line". Therefore if east is taken as positive direction, then west should be negative. Here the idea is illustrated with charts relevant to addition and subtraction. In these charts, zero is at the centre of negative and positive numbers.





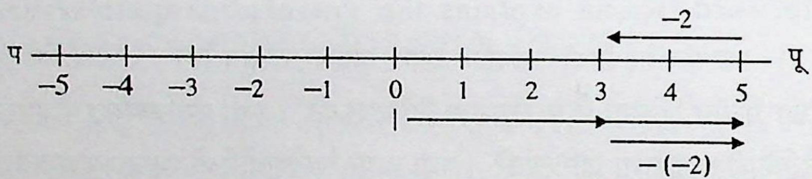
$3-(+2)=1$

Fig. 1



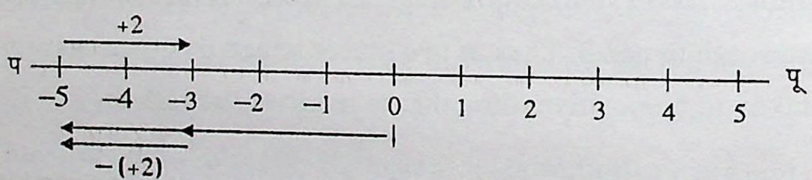
$-3-(-2)=-1$

Fig. 2



$3-(-2)=5$

Fig. 3



$-3-(+2)=-5$

Fig. 4

The preceding charts precisely illustrate Kṛṣṇa's statement below (*BP*. p. 15) : इदमेव प्रतीत्यर्थं पूर्वपश्चिमदेशत्वेन योज्यते । पू ३ पू २ संशोध्यमानः पूर्वदेशः पश्चिमदेशो भवतीति जातम् पू ३ पू २ । अनयोर्धनर्णयोरन्तरमेव योगः इति शेषमन्तरम् पू १ (Fig.1) । अत्रैकस्मादवधेः पूर्वतो योजनद्वयेन त्रयेण च नरौ तिष्ठतः । तत्र योजनद्वयगतात् योजनत्रयगोयोजनमेकं पूर्वतः तिष्टतीत्यर्थः । अत्रोदाहरणेषु द्वयस्य शोध्यतोक्तेः योजनद्वयगान्नरादन्तरं ज्ञातव्यम् ।

अथ द्वितीये प ३ पू २ । उक्तवदन्तरे जातं प १ (Fig.2) । पश्चिमतो योजनत्रयगतः पश्चिमतो योजनद्वयगतादेकेन योजनेन पश्चिमतस्तिष्टतीत्यर्थः ।

तृतीये न्यासः पू ३ प २ उक्तवदन्तरे जातं पू ५ । (Fig.3) पश्चिमतो योजनद्वयगतात् सः पूर्वतो योजनत्रयगः पू ५ पञ्चभिर्योजनैः पूर्वतस्तिष्ठतीत्यर्थः ।

चतुर्थे न्यासः पू ३ प २ उक्तवज्जातमन्तरं प ५ । (Fig.4) पूर्वतः योजनद्वयगतात् पश्चिमतः योजनत्रयगः पञ्चभिर्योजनैः पश्चिमतस्तिष्ठतीत्यर्थः ।

2.1.1. Symbol for a Negative Number :

Samśodya, apanīya are some of the words used for the operation of subtraction. Bhāskara also says (*BG.* p. 2) that the dot on top of the number shows that it is negative : यानि ऋणगतानि तानि ऊर्ध्वबिन्दूनि च । But Kṛṣṇa makes this apt observation (*BP.* p. 11) : अतिरोहितार्थमिदम् । यदृणत्वादिकं आलापत एव अवगन्तुं शक्यम् । – This is evident even if one can know whether a number is negative from the enunciation of the question itself. But he adds : तथाप्यालापबहुत्वे ऋणत्वादौ भ्रान्तिः संशीतिर्वा स्यादुपस्थितिलाघवं च न स्यादित्यूर्ध्वबिन्द्वादिलेखनं युक्ततरम् । – Even then, when there are many statements and there is likelihood of misunderstanding or doubt about negativeness, the placing of dot above the number will make it easier to understand the situation.

2.1.2 *Guṇana* - Multiplication :

The common Indian name for multiplication is *guṇana.* The multiplicand (that which is multiplied) is termed *guṇya* and the multiplier (that which multiplies) is *guṇaka* or *guṇakāra.* The product is *guṇanaphala. Vadhaḥ* and *ghātaḥ* are other terms that are used to indicate multiplication.

Brahmagupta writes of multiplication (*Br. Sp.* p.1189) : ऋणमृणधनयोर्घातः धनमृणयोर्धनवधो धनं भवति । – that is, the product of a positive and negative (number) is negative, of two negatives is positive ; positive multiplied by positive is positive.

Bhāskara explains the same in a simple *sūtra* (*BG.* v. 9) : स्वयोरस्वयोः स्वं वधः स्वर्णघाते क्षयः ।।

2.1.3. *Bhajana / Haraṇa* - Division :

For division Brahmagupta writes (*Br. Sp.* p.1193) : धन भक्तं धनम् ऋणहृतम् ऋणं धनं भवति। भक्तमृणेन धनमृणं धनेन हृतमृणमृणं भवति ।। – Positive divided by positive is positive and negative divided by negative is positive. But positive divided by negative and negative divided by positive is negative.

Again, Bhāskara simplifies this statement as (*BG.* v. 11) : भागहारेऽपि चैवं निरुक्तम् – The rule for multiplication applies for division also.[3]

2.1.4. Square roots of Negative Numbers :

The Indian Mathematicians were very well aware that a negative number has no square root as the rules mentioned just above make it clear. Mahāvīra says (*GSS.* v.52cd) : ऋणं स्वरूपतोऽवर्गो यतस्तस्मान्न तत्पदम् । – since a negative number by its own nature is not a square, it has no square root. Śrīpati declares : "A negative number by itself is a non-square. So its square root is unreal".[4]

Bhāskara says (*BG.* v. 13cd) : न मूलं क्षयस्यास्ति तस्याकृतित्वात् ।। – A negative number has no square root because of its non-square nature. Kṛṣṇa explains this concept quite clearly in terms of what was pointed out at the outset of this subsection. (*BP.* p. 19) : ऋणाङ्कं वर्गं वदता भवता कस्य स वर्ग इति वक्तव्यम् । न तावत् धनाङ्कस्य "समद्विघातो हि वर्गः" तत्र धनाङ्केन धनाङ्के गुणिते यो वर्गो भवेत् स धनमेव "स्वयोर्वधः स्वम्" इत्युक्तत्वात् नापि ऋणाङ्कस्य । तत्रापि समद्विघातार्थम् ऋणाङ्केन ऋणाङ्कगुणिते धनमेव वर्गो भवेत् "अस्वयोर्वधः स्वम् " इत्युक्तत्वात् । एवं सति कथमपि तमङ्कं न पश्यामो यस्य वर्गः क्षयः भवेत् । " – If it should be that a negative quantity may be a square, it should be shown what it is a square of. It cannot be the square of a positive

3. *Cf.* Georges Ifrah, *The Universal History of Numbers*, John Wiley and Sons Inc., 2000, 10th ed., p.439 : He wrongly translates the rule as "... The product or the quotient of two debts is one debt, ... "

4. Datta and Singh, *op.cit.*, Vol. II. p.24.

quantity because a square is the product of two like quantities ; and if a positive quantity is multiplied by another positive, the product is positive. Nor can it be a square of a negative quantity, for the product of two negatives is also positive. Therefore there cannot be any quantity such that its square is negative.

It is interesting to note the following remarks of Bourbaki in his book on *Theory of Sets*, which brings forth the fact that the West was struggling with the concept of negative numbers even in the 18th Century : "The embarassment of algebraists in the presence of negative numbers vanished only when analytical geometry provided them with a convenient 'interpretation'. But, even in the eighteenth century, d'Alembert (although a convinced positivist) when discussing the question in the Encyclopaedia, suddenly lost courage after a column of somewhat confused explanations, and contended himself with concluding that 'the rules of algebraic operations on negative quantities are generally admitted by everyone and are generally received as correct, whatever interpretation is to be attached to these quantities'".[5]

The above shows the clear understanding of the rule of signs in Indian mathematics comp ared to the confusion that prevailed on negative numbers in Europe even as late as the 18th Century.

2.2. *Khaṣadvidha*:

The discovery of zero by Indians not only helped to establish the place value system but also to distinguish between negative and positive numbers.

Georges Ifrah, observes in his book, *The Universal History of Numbers* (p. 597) : "For thousands of years people stumbled along with inadequate

5. Quoted by M.D. Srinivas, "The Methodology of Indian Mathematics and its contemperory relevance", *PPST Bulletin* 12 (Sept. 1987), p. 30.

useless systems which lacked a symbol for 'empty' for 'nothing'. Similarly there was no way of conceiving of 'negative' numbers (–1, –2, –3 etc.,), such as we nowadays use routinely to express for example, sub zero temperatures or bank accounts in deficit. Therefore, a subtraction such as '3–5' was for a long time considered impossible. The discovery of zero swept away this obstacle so that the ordinary "(natural)" numbers were extended to include this 'mirror images' with respect to zero".

This breakthrough was realised when Brahmagupta in his *Br.Sp.* gave rules for zero operations. By opening the way to the generalisation of the concept of the number, Ifrah says (p.439) : "the Indian scholars enabled the rapid development of algebra . . .".

Simple arithmetical operations involving zero were known and carried out in India earlier than the time of Brahmagupta. For instance, a statement in the *Pañcasiddhāntika* (IV.8) of Varāhamihira implies the addition 30 + 0 as *trimsat yuktāmbara* [*trimsat* (thirty) plus *ambara* (zero)]. But Brahmagupta seems to be the first in the world to give a formal exposition of *sūnyagaṇita* in his *Br. Sp.* (XIII.30). For addition involving positive, negative and zero numbers he states : धनयोर्धनमृणमृणयोर्धनर्णयोरन्तरं समैक्यं खम् । ऋणमैक्यं च धनमृणधनशून्ययोः शून्ययोः शून्यम् ।। – The sum of two positive numbers is positive, of two negative numbers is negative ; of a positive and a negative number is their difference; if they are equal, zero. The sum of a negative and zero is negative; of positive and zero is positive; of two zeros is zero. Ironically more than twelve centuries after Brahmagupta's correct rule $0 \times 0 = 0$, a French scholar by name Antoine L.G. Demonville[6] gave the wrong rule $0 \times 0 = 1$.

2.2.1. Operations of Zero :

Brahmagupta as said earlier, was the first to deal with operations of zero. He has enumerated them as (*Br. Sp.* XVIII. 32, 33cd, 34b, 35) :

6. Refer R.C. Gupta, "Zero in the mathematical system of India", *The Concept of Śūnya*, IGNCA and INSA, New Delhi, 2003, p. 157, fn. 6

शून्यविहीनम् ऋणम् ऋणं धनं धनं भवति शून्यमाकाशम् ।
शोध्यं यदा धनम् ऋणाद् ऋणं धनाद्वा तदा क्षेप्यम् ॥

– Negative less zero is negative ; positive less zero is positive ; zero less zero is zero. Positive less negative and negative less positive is their difference.

In *BG,* the same results are given with the addition that if a quantity is subtracted from zero, the sign is reversed (v. 16) :

खयोगे वियोगे धनर्णे तथैव च्युतं शून्यतस्तद्विपर्यासमेति ।

Kṛṣṇa explains that *khayoga* can be two-fold (*BP.* p. 24) : a) खेन योगो रूपादेः खयोग इत्येकः – addition of zero to a number is one case of zero-addition ; b) खस्य योगो रूपादिना खयोग इति द्वितीयः – addition of number to zero is another case of zero-addition. Similarly *khaviyoga* can also be two-fold : a) खेन वियोगः इत्येकः – subtraction of zero from a number is one case of zero subtraction ; b) खात् वियोगः इति द्वितीयः – subtraction of a number from zero is another case.

The example given here is (*BG.* v. 17) :

रूपत्रयं स्वं क्षयगं च खं च ।
किं स्यात् खयुक्तं वद खच्युतं च ॥

Kṛṣṇa gives mathematical illustrations for both addition and subtraction.

A number is termed *yojya* when it is added to, and *yojaka* when it adds.

Let *yojya* be 3, *yojaka* be 4

One can deduce that $3 + 0 = 3$ by the following steps just by using the fact that a number added to its negative is 0.

$$3 + 3 \Rightarrow 3 + (4 - 1) = 7 - 1 = 6$$
$$3 + 2 \Rightarrow 3 + (3 - 1) = 6 - 1 = 5$$
$$3 + 1 \Rightarrow 3 + (2 - 1) = 5 - 1 = 4$$
$$3 + 0 \Rightarrow 3 + (1 - 1) = 4 - 1 = 3$$

Calling the number to be subtracted, *viyojya* (5), and that which is subtracted as *viyojaka* (3) in the context of subtraction, the following steps lead to the rule about *khacyuta* in the sense of subtraction from zero.

$$5 - 3 = 2$$
$$4 - 3 = 2 - 1 = 1$$
$$3 - 3 = 1 - 1 = 0$$
$$2 - 3 = 0 - 1 = -1$$
$$1 - 3 = -1 - 1 = -2$$
$$0 - 3 = -2 - 1 = -3$$

Kṛṣṇa adds finally (*BP*.p.26) इति उपपन्नं च्युतं शून्यतस्तद्विपर्यासमेति । – Thus it has been proved that when zero is added to or subtracted from, any number, the number remains the same.

Note : The steps which precede precisely extend the table of subtraction by 3 beyond the level $3 - 3 = 0$ motivating the definition of $2 - 3$, $1 - 3$, $0 - 3$ etc.

2.2.2. Zero as infinitesimal :

The idea of zero as an infinitesimal is quite evident from Bhāskara's *Līlāvatī* [7] (v. 46c) : खगुण: खम् । – The product of (a number) and zero is zero.

7. All references to the text *Līlāvatī* are from the edition of the text by Motilal Banarasidass, New Delhi, 2001.

Kṛṣṇa proves the result (*BP.* p. 27) : $a \times 0 = 0 = 0 \times a$.

गुण्यस्य परमापचये गुणनफलस्यापि परमापचयेन भाव्यम् । परमापचये च शून्यतैव पर्यवस्यतीति शून्ये गुण्ये गुणनफलं शून्यमेवेति सिद्धम् । . . . एवं गुणकापचयवशादपि गुणनफलेऽपचयात् गुणकस्यापि शून्यत्वे गुणनफलशून्यमेवेति सिद्धम् । – The more the *guṇya* is diminished, the smaller is the product; and if it is reduced to the utmost degree, the product gets reduced likewise ; now the utmost reduction of a quantity is equivalent to the reduction of it to nothing ; therefore, if *guṇya* is zero, the product is zero . . . In like manner, as the multiplier, *guṇaka* decreases, so does the product; and if the *guṇaka* is zero, the product also is zero.

Note : In the above discussion the *guṇya* and *guṇaka* are assumed to be positive integers, to begin with, leading to the fact that $0 \times a = 0$ for any positive integers. It is evident that in the above, zero is conceived of as the least of the set of all non-negative integers.

2.2.3. Kṛṣṇa on *khahara*:

Similarly Kṛṣṇa explains (*BP.* p. 28) – the evolution of *khahara* $\frac{a}{0} = \infty$ from the fraction $\frac{a}{b}$:

यथा यथा भाजकस्यापचयः तथा तथा लब्धेरुपचयः तथा सति भाजके परमापचिते लब्धेः परमोपचयेन भाव्यम् । लब्धेश्चेत् इयत्तोच्येत तर्हि परमत्वं न स्यात्ततो अपि आधिक्यसंभवात् । अतो लब्धेरियत्ताभाव एव परमत्वं तदेवोपपन्नं खहरो राशिः अनन्त इति । – "As much as the divisor (b) is diminished, ($a,b > 0$ tacitly) so much is the quotient $\left(\frac{a}{b} \mid a,b > 0\right)$ increased. If the divisor is reduced to the utmost (*parama*) the quotient is increased to the utmost. But if it can be specified, that the amount of quotient is so much, it has not been raised to the utmost because a greater number than that is possible. Therefore the quotient is great without limit, and the resulting *khahara* quantity is infinite (*ananta*).[8]"

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8. R.C. Gupta, *op.cit.*, p.156.

Note : The symbols a, b have been assumed to be positive integers in the above observation. It implies that the Indian mathematicians were aware of the limit $\lim_{\sigma \to 0} \frac{N}{\sigma} = \infty$ (numerically), for non zero N. It may be pointed out here that $\propto$ is not a number but the relation just signifies that $\frac{N}{\sigma}$ becomes numerically great when σ becomes numerically small but not equal to zero.

To establish the fact that $\infty + a = \infty$ (infinity plus a number = infinity) and $\infty - a = \infty$ (infinity minus a number is infinity), Bhāskara says (*BG.* v. 20) :

अस्मिन् विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु ।
बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत् ।।

— Just as at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth, there is no change found in the Infinite and Immutable God, so in the quantity which has zero for its divisor, there is no change, though many may be added or subtracted.

Later authors like Gaṇeśa, author of *Gaṇitamañjarī* says R.C. Gupta[9], have argued that Bhāskara's contention $\infty \pm a = \infty$ i.e., there is no change in infinity when a number is added to it or subtracted from it, is wrong. For, $4/0 \pm 11/2 = 8/0$ by the usual arithmetical process. Thus $4/0$ has changed to $8/0$ and Gaṇeśa concluded that there is a change in the *khahara*. However Kṛṣṇa rules out such a criticism pointing out that though forms like $4/0$, $8/0$ look different, their meaning is the same.

It is interesting to note that Kṛṣṇa was aware of the conceptual thinking in these symbols as understood in recent times and called attention to them (*BP.* p. 29) : अत्रापि फलतो विकाराभावात् । न हि खेन भक्तेषु त्रिष्वन्यत्फलम् अष्टसु भक्तेष्वितरदिति । किं तु उभयत्राप्यनन्तत्वं न व्यभिचरति ।

9. *loc.cit.*

To explain the above, Kṛṣṇa has taken an example from astronomy. The length of the shadows of a gnomon of length 'g' when the sun's altitude is α is given by the formula $S = g\,(R \cos \alpha)/(R \sin \alpha)$ where R is the radius of the circle of reference (chosen arbitrarily). At sunrise, the altitude α is zero and the shadow will be given by $S = \frac{gR}{0}$. The usual value of 'g' is 12 *aṅgulas.* For a few values of R, Kṛṣṇa correctly gets corresponding values of S as follows :

R	3438	120	100	90
S	41256/0	1440/0	1200/0	1080/0

Since, even with the choice of different radii, the length of the shadow of a gnomon at any time must be the same (*chāyā tulyaiva*), Kṛṣṇa concludes that the above different values of the shadow must all be equal. This illustrates that $\frac{a}{0} = \frac{b}{0}$. Thus each of the above represents the same *khahara* - infinity (*BP.* p. 30) :

किं च उन्नतांशजीवास्वरूपे शङ्कौ यदि दृग्ज्याभुजस्तदेष्टे द्वादशाङ्गुलादिके शङ्कौ किमिति त्रैराशिकेन च्छाया सिध्यति । तत्रोदयकाले उन्नतजीवाया अभावः दृग्ज्या च त्रिज्यामिता १२० अत्र द्वित्रिचतुरङ्गुलादीनां शङ्कुना मुक्तः त्रैराशिकेन छायासाधने २४० । ३६० । ४८०

. . . एतदाद्याः सिद्ध्यन्ति खहराः छायाः न ह्येतासु फलतो वैलक्षण्यमस्ति । यतस्तस्मिन्नपि काले न्यूनाधिकपरिमाणानामपि शङ्कूनां छायानन्त्यं न व्यभिचरति । . . . किं तु नानात्रिज्याभ्योनुपातसिद्धा छाया तुल्यैवेति सकलगणकानामविवाद इति सर्वमवदातम् ।

2.3. *Avyaktaṣaḍvidha* :

All six operations in connection with unknowns fall under this category. However, here, the terminology of *avyakta,* the process of division and finding the square root with which Kṛṣṇa deals in detail, alone are presented.

2.3.1. Symbols for *avyaktas* (unknowns) :

a) The *avyaktas* were known in the Jaina text, *Sthānāṅga sūtra*[10] (before 300 B.C.) as *yāvat tāvat.*

b) Brahmagupta mentions *varṇa* as symbols for unknowns (*Br.Sp.* XVIII. 2, 42, 51).

c) यावत् तावत् च साकल्येऽवधौ मानेऽवधारणे says Amarasimha in his *Amarakośa*[11]. To him *yāvat tāvat* means a measure or quantity, maybe an unknown quantity.

d) T. Hayashi in his edition of *BM*[12] discusses the term as follows :

"The first definite testimony for the term *yāvat tāvat* in the sense of unknown quantity occurs in *Āryabhaṭīyabhāṣya* of Bhāskarācārya I. While explaining an example, Bhāskara says -

एत एव गुलिका अज्ञातप्रमाणा यावत् तावन्त उच्यन्ते । . . .

In another place Bhāskara refers to *yāvat tāvat* as one of the four *bījas* which generate the eight kinds of *vyavahāra* or practical problems :

. . . चत्वारि बीजानि प्रथमद्वितीयतृतीयचतुर्थानि यावत्तावद्वर्गावर्गघनाघनविषमानि . . ।

Here *yāvat tāvat* has been used in the sense of equations with one unknown. It is probably significant that Brahmagupta who was a contemporary of Bhāskara I, never uses the word *yāvat tāvat* for unknown but instead uses the terms *avyakta* or *varṇa*". Hayashi concludes by saying ". . . I am inclined to conjecture that originally the 'colours' belonged to one tradition and the *yāvat tāvat* to another".

10. Datta and Singh, *op.cit.*, Vol. II. p.9.

11. *Amarakośa*, ed. A.A. Ramanathan, The Adyar Library & Research Centre, 1978, p. 587.

12. T. Hayashi, *op.cit.*, pp. 78-81.

e) Bhāskara says (*BG.* v. 21) :

यावत्तावत् कालको नीलकोऽन्यो वर्णः पीतो लोहितश्चैतदाद्याः ।
अव्यक्तानां कल्पिता मानसंज्ञास्तत्संख्यानं कर्त्तुमाचार्यवर्यैः ।।

Bhāskara thus clearly says that in algebra, the initial letters of (the names of) knowns and unknowns should be written for implying them. Thus in the text – i) '*yā*' is used for *yāvat tāvat*, '*kā*' for *kālaka*, '*nī*' for *nīlaka* and so on; ii) initial letters or syllables of words also signify the subjects of the problem, such as '*mā*' for *māṇikya*, '*mu*' for *muktāphala* and so on; iii) names of geometric lines in geometric figures such as '*bhu*' for *bhuja*, '*ko*' for *koṭi.*

It is evident that *yāvat tāvat* is not a *varṇa* like *kālaka, nīlaka* and so on. So in its inclusion, Datta and Singh say that : "we find persistence of an ancient symbol which was in vogue long before the introduction of colours to represent unknowns".[13]

f) The product of an unknown *yāvat* with another *yāvat* was called *yāvat tāvat varga*. But the product of *yāvat tāvat* with another unknown say *kālaka* was called *bhāvita.*

Brahmagupta calls the product of two unknowns as *bhāvitaka* (Br.Sp.XVIII.42 cd):

अन्योऽन्यवर्णघातः भावितकः पूर्ववच्छेषम् ।

But Bhāskara calls it *bhāvita* (*BG.* v. 26cd) :

वधे तु तद्वर्गघनादयः स्युः तद्भावितं च असमजातिघाते ।

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13. Datta and Singh, *op.cit.*, Vol. II, p.20 ; also, Mm. Muralidhara Jha has an interesting observation to make regarding *yāvat tāvat.* In the preface (p. 2) to his edition of Bhāskara's *BG,* he emends the term *yāvat tāvat* as *yāvakastāvat* wherein *yāvaka* means red colour: रक्तवर्णो यावकः । नामैकदेशे नामग्रहणं इत्यतो यावः । तत्र तावत् प्रथममव्यक्तराशेर्मानं याव इति "यावस्तावतं कल्प्यमव्यक्तराशेः" अत्र 'स्ता' इत्यक्षरस्य कश्चित् भागः कालदोषान्नष्टोऽतो "यावत्तावत्" संप्रति प्रसिद्धः । वस्तुतोऽयं पाठो भास्करसमयादेव विकृतः ।

In Indian mathematics, *vadha, ghāta, hara, guṇa* are various words used for multiplication. Kṛṣṇa considers these words in the place of *bhāvita*, and rejects them before accepting *bhāvita* (*BP.* p. 41) :

तस्मादसमजातिघाते तयोर्घात इत्यक्षरतो भवितुं युक्तम् । तत्राद्यैर्घातस्य भावितमिति संज्ञाकृता वधशब्दस्य आद्याक्षरलिखने यावदादिवर्गेण सङ्करः स्यात्। घातशब्दस्याद्यक्षरलिखने कदाचिद्घनेन सङ्करः स्यात् । गुणशब्दस्य प्रथमाक्षरलिखने अश्लीलता स्यात् । हति शब्दप्रथमाक्षरलिखने कदाचिद्धरितकवर्णभ्रमः स्यादिति । अथ यद्यपरः कश्चिच्छब्दोऽस्ति यत्प्रथमाक्षरलिखने सङ्करादिदोषो न स्यात् । अस्तु तर्हि तल्लिखनं न काचित्क्षतिः । किन्तु आचार्येणाद्यानुरोधात् भावितमिति संज्ञाकृता । – the first letter is used to denote multiplication of different unknowns. To denote it, the word *bhāvita* has been chosen. If *'va'* as in *vadha* is used then it could be mistaken for *varga* or square ; if *'gha'* is used, it could sometimes mean *ghana* or cube.The use of *'gu'* would be vulgar. Similarly the use of *'ha'* as in *hara* would wrongly imply *harita,* another unknown. Therefore any other word where first syllable is suitable can be used ;there is no harm in doing so. Hence the word *bhāvita* is chosen by the Ācāryā.

Note : The above discussion brings out the attempts of algebraists' practical problem with choice of algebraic symbols reflecting the nature of the entities for which they stand.

2.3.2. Methods of Division :

In the chapter on *avyakta ṣaḍvidha* dealing with unknowns, after explaining the methods of addition, subtraction and multiplication, Bhāskara leaves methods of division to the students.

Kṛṣṇa however takes this up and explains fully how the division operation is to be done . He works out one of the examples of Bhāskara, taking the product of the former example to be the dividend, one factor as its divisor and arrives at the other factor as the quotient (*BP*.p.48): – अथेदं गुण्यभक्तं किं स्यादिति भागहारार्थं गुण्यछेदस्य गुणनफलस्य न्यासः याव १८ याकाभा २४ यानीभा १२ं या १२ं काव ८ कानीभा ८ं का ८ं नीव २ नी ४ रू २.

The example given is presented in modern notation.

$$\frac{18x^2 + 24xy - 12xz - 12x - 12y^2 - 8yz - 8y + 2z^2 + 4z + 2}{-3x - 2y + z + 1}$$

Divide $18x^2$ by $-3x$, the quotient is $-6x$; $-6x$ multiplied by $-3x - 2y + z + 1$ gives $18x^2 + 12xy + \; - 6xy - 6$.

Subtract this from the quantity to be divided and find out the quotient which would be equal to the leading term in the remainder when multiplied by $(-3x)$. Proceed similarly as shown below till all the terms in the dividend are exhausted.

$$\begin{array}{l}
\quad\quad\quad\quad\quad\quad\quad\;\; \underline{-6x - 4y + 2z + 2} \\
-3x - 2y + z + 1 \,\big)\; 18x^2 + 24xy - 12xz - 12x + 8y^2 - 8yz - 8y + 2z^2 + 4z + 2 \\
\quad\quad\quad\quad\quad\quad\quad\;\; \underline{18x^2 + 12xy - 6xz - 6x} \\
\quad\quad\quad\quad\quad\quad\quad\quad\; 12xy - 6xz - 6x + 8y^2 - 8yz - 8y + 2z^2 + 4z + 2 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; \underline{12xy \quad\quad\quad\quad + 8y^2 - 4yz - 4y} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; - 6xz - 6x \quad - 4yz - 4y + 2z^2 + 4z + 2 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{- 6xz \quad\quad\quad - 4yz \quad\quad + 2z^2 + 2z} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad - 6x \quad\quad - 4y \quad\quad + 2z + 2 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{- 6 \quad\quad - 4y \quad\quad + 2z + 2} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; \underline{\quad\quad 0 \quad\quad\quad\quad}
\end{array}$$

The method described above is very much the same as is used in modern times.

2.3.3. Squares and Square Roots :

Kṛṣṇa also details the method of finding the squares of unknowns and their square root. He takes a simple example $16x^2 - 48x + 36$. Following the rule (*BG.* v. 31) :

कृतिभ्य आदाय पदानि तेषां द्वयोर्द्वयोश्चाभिहतिं द्विनिघ्नीम् ।
शेषात् त्यजेद्रूपपदं गृहीत्वा चेत्सन्ति रूपाणि तथैव शेषम् ॥

we get the square root of $16x^2$ as $4x$ and 36 as 6. Now $4x$ multiplied by 6 gives $24x$; since the example has $(-48x)$ the square root should be $4x-6$ and not $4x+6$. The square of $4x+6$ would be $16x^2+48x+36$.

2.4. *Karaṇīṣaḍvidha* (Surds) :

The word *karaṇī* is probably derived from the word *karaṇa* which means 'doing', 'producing' and hence 'that which makes'.

2.4.1. The term *karaṇī*:

In the *Kātyāyana Śulba sūtras* (II.3), *karaṇī* meant "the cord used for measuring of a square". Datta and Singh remark[14] that *karaṇī* denotes a square root in the *Śulba sūtras* and Prakrit literature. In *kṣetra-gaṇita* or geometry it denotes a side. In later times, the term denotes a surd, i.e. a square root which cannot be evaluated. The term surd is used for simplicity, here.

Defining *karaṇī* Śrīpati says (*Siddhānta śekhara* XIV.7 ab) :

ग्राह्यं न मूलं खलु यस्य राशेस्तस्य प्रतिष्टं करणीति नाम ।

that it is an irrational square root.

T. Hayashi in his edition of *BM* (pp. 63-4) traces the term *karaṇī* elaborately : "Gaṇeśā, a famous scholiast on *Līlāvatī* explains the term thus :

यस्य मूले गृह्यमाणे संयंमूलं न लभ्यते तन्नाम करणी । तदुपचाराद् वर्गराशेरपि मूले गृह्यमाणे करणीति उच्यते । पूर्वेषां पारिभाषिकायां संज्ञा । तथा चाहुः – मूलं ग्राह्यं राशेस्तु यस्य करणीति नाम तस्य स्यादिति । – "A certain number whose square root is

14. Datta and Singh, *op.cit.*, Vol. I, p. 170.

being extracted but cannot be obtained exactly is called *karaṇī.* In its secondary application, even a square number, when its square root is being extracted is called *karaṇī.* This is a technical term of our predecessors. Thus they say - the name of the number whose square root is to be taken is *karaṇī*'' . . . the term was, in mathematical works, used atleast in five different senses . . .

1. Making (a square figure) in the *Śulba sūtras,*

2. 'The square power' or a 'number in the square power' in the *Pañcasiddhāntika*, in Bhāskara's *Āryabhaṭīyabhāṣya*, in the *Sūrya-siddhānta*, in the *Gaṇitatilaka*, and in the *BM.*

3. 'The square root'

a) The square root of a non-square number, in the Prākṛt sources cited in Bhāskara's *Āryabhaṭīyabhāṣya*, in the *Āryabhaṭīyabhāṣya* itself and in the *BM.*

b) 'The square root in general' in the Sanskrit sources cited in the anon.comm. on Śrīdhara's *Pāṭigaṇita.*

4. 'A number square or non square whose root is required' in the *Brahmasphuṭasiddhānta,* in the *Āryabhaṭīyabhāṣya,* in the *GSS*, in the Sanskrit sources cited in anononymous commentary on Śrīdhara's *Pāṭigaṇita,* in the *Siddhānta śekhara,* in Bhāskara's *Bījagaṇita* and in Gaṇeśā's *Buddhavilāsinī* on the *Līlāvatī.*

5. 'The greatest common divisor' in the *Mahāsiddhānta*''.

Kṛṣṇa explains the term *karaṇī* succinctly as (*BP.* p. 50) : तत्र यस्य राशेर्मूलेऽपेक्षिते निरग्रं मूलं न संभवति स करणी । – Here, the number for which a square-root is needed but cannot be evaluated, is a *karaṇī.*

Note : In the above remarks, the term, 'cannot be evaluated' stands for 'not a rational number' in the modern sense. Even today, in Indian languages, the term *karaṇī* or synonymously *karṇam* is used for hypotenuse of a right-angled triangle. In the *Śulba sūtras* $\sqrt{2}$ appeared as the *karṇam* of an isosceles right triangle of unit equal sides. It was noted that this *karṇam* could not be evaluated (as a rational number). This *karṇam* is again a displacement from one point to another by a process of going one unit in a direction and one more unit in a perpendicular direction. Thus $\sqrt{2}$ which appeared as a first instance by which a process (*karaṇīyam*) was called *karaṇī* and the name was extended to all numbers with the similar characteristic. It may be observed that these are cases of 'a surd' in modern terminology.

2.4.2. Addition of *Karaṇī* :

For addition of *karaṇī*, Bhāskara gives the following *sūtra* (*BG.* v. 34) :

योगं करण्योर्महतीं प्रकल्प्य घातस्य मूलं द्विगुणं लघुं च ।
योगान्तरे रूपवदेतयोस्ते वर्गेण वर्गं गुणयेद् भजेच्च ॥

— The sum of two numbers under the square root sign is denoted by M (*mahatī*). Twice the square root of their product is denoted by L (*laghu*). The sum and difference of the two surds are respectively $\sqrt{M+L}$ and $\sqrt{M-L}$. If a surd is to be multiplied or divided by a given number, multiply or divide the number under the radical sign by the square of the given number.

Hankel says appreciatively that : "In Bhāskara, we find two remarkable identities one of which is given in nearly all our school algebras, as showing how to find the square root of a 'binomial surd'. What Euclid in Book X embodied in abstract language, difficult of

comprehension, is here expressed to the eye in algebraic form and applied to numbers.''

The rule is, $a + b = M$, $2\sqrt{ab} = L$

then $\left(\sqrt{a} \pm \sqrt{b}\right)^2 = a + b \pm 2\sqrt{ab}$

therefore $\sqrt{a} \pm \sqrt{b} = \sqrt{M \pm L}$

2.4.3. Division of *Karaṇī* :

Bhāskara after explaining in detail the operations of addition, subtraction and multiplication of surds gives two methods for division. Since straight division method for surds may be difficult, Bhāskara gives an alternate method. Kṛṣṇa with great reverence says that the Ācārya has given the alternate method for the easy solution of the problem (*BP.* p. 59): अत्र द्वितीयोदाहरणे भाजकः कियद्गुणः भाज्यात् शुध्यतीति दुरवबोधम् । अतः परमकारुणिकैराचार्यैः शिष्यबोधार्थमुपायान्तरमुपजातिकाद्वयेन निरूप्यते ।

What now Kṛṣṇa goes on to explain in detail is in modern mathematics called the method of rationalising the denominator of a surd. The problem on hand is division of $\sqrt{300} - \sqrt{256}$ by $\sqrt{27} - \sqrt{25}$. The method is as follows :

Since the denominator is $\sqrt{27} - \sqrt{25}$, multiply and divide by $\sqrt{27} + \sqrt{25}$ where $\sqrt{25}$ has a positive sign.

$$\frac{\sqrt{300} - \sqrt{256}}{\sqrt{27} - \sqrt{25}} \times \frac{\sqrt{27} + \sqrt{25}}{\sqrt{27} + \sqrt{25}}$$

Numerator would be $\left(-\sqrt{256} + \sqrt{300}\right)\left(\sqrt{27} - \sqrt{25}\right) =$

$-\sqrt{6912} + \sqrt{7500} + \sqrt{8100} - \sqrt{6400}$;

Adding $\sqrt{8100} - \sqrt{6400}$, $\sqrt{7500} - \sqrt{6912}$ we get $\sqrt{100} + \sqrt{12}$.

Denominator would be $\left(\sqrt{27}-\sqrt{25}\right)\left(\sqrt{27}+\sqrt{25}\right) = 27 - 25 = \sqrt{4}$.

$$\therefore \text{ Quotient } = \frac{\sqrt{100}+\sqrt{12}}{\sqrt{4}}$$

$$= \sqrt{25}+\sqrt{3}$$

In fine, the above analysis on the six fundamental operations in algebra, forming the first six chapters of *BP*, brings out the distinct views of Kṛṣṇa on the same.

CHAPTER - 3
KUṬṬAKA — LINEAR EQUATIONS OF FIRST DEGREE

After giving the definition of the word *kuṭṭaka* and explaining the method, Bhāskara has considered various forms of *kuṭṭaka* with negative and positive dividend, divisor and additive. Kṛṣṇa adds his own innovations and short methods in solving the linear equation in two unknowns. Some errors of previous authors pointed out by Kṛṣṇa are also highlighted. Two special kinds of *kuṭṭaka,* the *Sthirakuṭṭaka* and *Samśliṣṭakuṭṭaka* are also taken for discussion. These special kinds are of use in astronomy as exemplified by Kṛṣṇa and detailed here.

3. 1. *Kuṭṭaka* :

L.E. Dickson in his *History of Theory of Numbers* says on the development of *kuṭṭaka* (a method to solve indeterminate equations of the first degree), as follows :

"An account of the method of solving $ax + by = c$ (was) given by the Hindu Brahmagupta in the seventh century. It was based on mutual division of a and b as in Euclid's process of finding their greatest common divisor. Essentially the same method was rediscovered in Europe by Bachet de Meziriac in 1612, and expressed in the convenient notation of the development of a/b into a continued fraction by Saunderson in England in 1740 and by Lagrange in France in 1767. The simplest proof that the equation is solvable when a and b are relatively prime is that given by Euler in 1760".[1]

The earliest Indian algebraist however, to give a treatment of the indeterminate equation of the first degree is Āryabhaṭa. Indeterminate

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1. L.E. Dickson, *History of Theory of Numbers*, Vol. II, Chelsea Publishing Co., New York, 1952, p.v.

equation of the first degree was considered important by Indian mathematicians. Beginning with Āryabhaṭa most of them dealt with the *kuṭṭaka* or pulverizer method to solve the above equations. These methods were also considered important in astronomy in finding the position of the planet and so on. *Kuṭṭaka* method is also used in solving some equations of several unknowns.

3.2. *Kuṭṭaka* defined :

The word *kuṭṭaka* is derived from the root कुट्ट 'to grind'. Hence 'pulverizer'. *Kuṭṭayati* means *cūrṇayati*, i.e., 'grinds'. Kṛṣṇa, following Bhāskara and Brahmagupta, makes out that it is natural that *kuṭṭaka* belongs to *avyaktagaṇita* (*BP.* p.85) : ''पाद्या च बीजेन च कुट्टकेन वर्गप्रकृत्या च तथोत्तराणि'' इति प्रश्नाध्याये कुट्टकस्य पृथङ्निर्देशात् ''परिकर्म विंशतिं यः संकलिताद्यां पृथग्विजानाति । अष्टौ च व्यवहारान् छायान्तान् भवति गणकः सः'' ।। इति ब्रह्मगुप्तादिपाटीगणितारम्भे पाटीस्वरूपकथनेऽनिर्देशात् च . . . । न तस्य व्यक्तान्तर्भूतत्वम् ।

— though it is said above that Bhāskara does not include it in arithmetic and algebra as quoted from his *praśnādhyāya* and also Brahmagupta does not include it in arithmetic as seen from his text, that since it is essential for solving equations of several unknowns (*anekavarṇa samīkaraṇa*), it surely belongs to *avyaktagaṇita.*

Note : What is meant is that just reversal of arithmetical operations would not help in solving these linear equations in two variables. Possibly, Āryabhaṭa and later authors had a method of justifying the process of *kuṭṭaka* and formation of *vallī* therefrom to get the solution which was not recorded by their followers. This justification needs to be algebraic and hence the nomenclature.

What is *Kuṭṭaka* ? Kṛṣṇa says (*BP.* p.86) :

a) कुट्टको नाम गुणकः । It is a kind of multiplier ;

b) कश्चिद् राशिः येन गुणितः उद्दिष्टक्षेपयुत ऊन उद्दिष्टहरेण भक्तस्सन् निश्शेषो भवेत् स गुणः कुट्टकः इति पूर्वेषां व्यपदेशात् । – earlier writers have defined *kuṭṭaka* as that unknown number (multiplier) which when multiplied by a certain number and added to or subtracted by an additive, when divided by a certain number leaves no remainder.

Note : Not only was one of the unknowns in the equation called *kuṭṭaka*, but the equation itself is called *kuṭṭaka* by Bhāskara and others.

3. 3. Method of *Kuṭṭaka* :

There are slight differences between the methods of Bhāskara and those of Āryabhaṭa I, Brahmagupta, Śrīpati and Āryabhaṭa II.

Kṛṣṇa says that the *kuṭṭaka* method is being explained by Ācārya to solve problems (*anekavarṇasamīkaraṇam*) involving several unknowns (*BG.* vv. 56-9) :

भाज्यो हारः क्षेपकश्चापवर्त्यः केनाप्यादौ संभवे कुट्टकार्थम् ।
येन छिन्नौ भाज्यहारौ न तेन क्षेपश्चैतद्दुष्टमुद्दिष्टमेव ॥

परस्परं भाजितयोर्ययोर्यः शेषस्तयोः स्यादपवर्तनं सः ।
तेनापवर्तेन विभाजितौ यौ तौ भाज्यहारौ दृढसंज्ञकौ स्तः ॥

मिथो भजेत्तौ दृढभाज्यहारौ यावद्विभाज्ये भवतीह रूपम् ।
फलान्यधोऽधस्तदधो निवेश्यः क्षेपस्तथाऽन्ते खमुपान्तिमेन ॥

स्वोर्ध्वे हतेऽन्त्येन युते तदन्त्यं त्यजेन्मुहुः स्यादिति राशियुग्मम् ।
ऊर्ध्वो विभाज्येन दृढेन तष्टः फलं गुणः स्यादपरो हरेण ॥

The process of solution is also called *kuṭṭaka*, and the various steps of *kuṭṭaka* is illustrated by Kṛṣṇa by the solution of the following example (*BP.* pp. 111-12) :

Solve for x and y in equation $ax + c = by$.

<u>Example :</u> $221x + 65 = 195y$

Following the definition of *dṛḍhasamjñā* as given by Gaṇeśa Daivajña[2], Kṛṣṇa says (*BP.* p.86) : दृढा इत्यन्वर्थसंज्ञा पुनः न अपवर्तते न क्षीयन्त इत्यर्थ इति व्याख्यावद्भिः श्रीगणेशदैवज्ञचरणैः उक्त एवायमर्थः । – that, if the *bhājya* (dividend), *hāra* (divisor) and *kṣepa* (additive) are divided by the greatest common divisor and brought to their lowest form, then they are called *dṛḍhasamjñā* ; the equation can then be solved by *kuṭṭaka* method.

Kṛṣṇa then gives an elaborate exposition of the method of finding the *apavartana aṅka* (greatest common divisor) and making both the *bhājya* (dividend) and *hāra* (divisor), *dṛḍha*. The *paraspara bhājana* method is very nearly the same as what is done in modern times to find the greatest common divisor of two numbers.

<u>Step 1</u>(p.111.line 3-7) : स्पष्टोर्थः ।। न्यासः भा २२१ / ह १९५ क्षे ६५ अत्रापवर्ताङ्क-ज्ञानार्थं भाज्ये २२१ हरेण १९५ भक्ते शेषं २६ अनेन पूनहरे भक्ते शेषं १३ अनेनापि पुनः पूर्वशेषे २६ भक्ते शेषाभाव : । अतः परस्परं भाजितयोरन्त्यशेषमिदं १३ इदमेव तयोरपवर्तनम् । अनेन तौ निश्शेषं भज्येते एव । अनेनापवर्तिता भाज्यहारक्षेपा जाता दृढाः । Before finding the solution, the *bhājya* (dividend) 221, the *hāra* (divisor) 195 and the *kṣepa* (additive) 65, have to be made *dṛḍha,* by being divided by a common divisor. By *paraspara bhajana* (mutual division),the *apavartanāṅka* (GCD) is found to be 13, i.e., *a, b, c* are made relatively prime.

```
195)221(1
    195
    ---
     26)195(7
        182
        ---
         13)26(2
            26
            --
             -
            --
```

2. *Līlāvatī* of Bhāskara, Pt. II with the com. *Buddhavilāsini* of Gaṇeśa Daivajña, ed. V.G. Apte, Anandasrama Sanskrit Series 104, Pune, 1859, p. 253.

Note : When the *bhājya* and *hāra* have no common fact or the problem is *duṣṭa,* in the sense that integer solutions do not exist.

Step 2 : (p.111, line 8) The equation then reduces to $17x + 5 = 15y$. Solving this equation is the same as solving the given equation. The dividend 17 is again divided by the divisor 15 till remainder 1 is reached.

Step 3 : (p.111, line 8-10)अनयोदर्दृढभाज्यहारयो: परस्परं भक्तयो: लब्धम् अधोधस्संदध: क्षेप: तदध: शून्यं निवेश्यम् इति जाता वल्ली । – The quotients thus obtained are placed one below the other and then the *kṣepa* and zero below it ; the *vallī* (columns)is thus formed.

$$
\begin{array}{l}
15)\ 17\ (1 \\
\quad\ \underline{15} \\
\quad\ 2)\ 15\ (7 \\
\qquad\ \underline{14} \\
\qquad\ \ \underline{1}
\end{array}
\qquad\qquad
\begin{array}{c}
\textit{Vallī} \\
1 \\
7 \\
5 \\
0
\end{array}
$$

Step 4 : (p.111, line 11)अत्र उपान्तिमेन स्वोर्ध्वे हते अन्त्येन युक्त अन्त्यं त्यजेत् । – The penultimate number of the *vallī* is multiplied by the number above and added to number below. This is written down and last number is discarded:

1	1
7	35
5	5
0	

Step 5 : (p.111, line 12) The process is repeated till two numbers alone are reached :

1	1	40
7	35	35
5	5	
0		

Step 6 : (p.111,line 13)एतौ दृढभाज्यहाराभ्याम् आभ्यां तष्टौ शेषे जातौ क्रमेण b pãY JUmi E – these two, *labdhi* (40) and *guṇa* (35) are to be respectively divided by *dṛḍhabhājya* (17) and *dṛḍhahāra* (15). The remainders 6, 5 are solutions for *labdhi* (y) and *guṇa* (x) respectively.

Note : $x = 35$, $y = 40$ are solutions; but step 6 is aimed at obtaining the smallest positive solution.

Bhāskara has added a few more corrollaries. They are :

3.3.1. To get infinite solutions :

Following the *sūtra* (*BG.* v. 67cd) : इष्टाहतस्वस्वहरेण युक्ते ते वा भवेतां बहुधा गुणाप्ती ।। – the *labdhi* and *guṇa* added to their respective divisors multiplied by assumed numbers become manifold, i.e. solutions for y and x ,

$$\left.\begin{aligned} y &= 17t + 6 \\ x &= 15t + 5 \end{aligned}\right] \quad t = 0, \ \pm 1, \ \pm 2, \ldots$$

Note : Though the *kuṭṭaka* is generally used to get the least positive solutions, it is evident from the *sūtra*, that there are infinite values of 't', some of which *labdhi* and *guṇa* can also be negative.

3.3.2. To find the solutions for negative *kṣepa* :

The *sūtra*, *BG.* v. 62ab, adds : योगजे तक्षणाच्छुद्धे गुणाप्ती स्तो वियोगजे । – the values of *guṇa* and *āpti* (=*labdhi*) obtained for positive *kṣepa* can be converted to values for negative *kṣepa* by reduction from *bhājya* and *hāra* respectively.

Example : To obtain solution set for $17x - 5 = 15y$

$17x + 5 = 15y$ has solution set (5, 6) i.e. $17(5) + 5 = 15(6)$

$17(5) + 2 \times 5 - 5 = 15(6)$, i.e. $17(5) + (17 - 15) \times 5 - 5 = 15(6)$

i.e. $17(5+5) - 5 = 15(6+5)$

Rewriting $17(5+5) - 5 = 15(6+5)$ i.e. $17(5) + 17(5) - 5 = 15(6) + 15(5)$

i.e. $17(15 - 5) - 5 = 15(17 - 6)$

i.e. $x = 10$, $y = 11$ are the solutions..

Note : When either the *bhājya* or *hāra* is negative, the *vallī* will have negative terms. This will be seen in some examples given later.

3.3.3. To find the solutions when *bhājya* and *kṣepa* alone are reduced :

Following Bhāskara's *sūtra* (*BG.* v. 65) Kṛṣṇa explains (*BP.* p. 109): अथवा भाज्यक्षेपौ द्वावपि हरेण तक्ष्यौ । तष्टयोः क्षेपभाज्ययोः प्राग्वदेव गुणाप्ती साध्ये । अत्र गुण एव ग्राह्यो न लब्धिः । – If the *bhājya* and *kṣepa* can both be reduced by the divisor or its multiple, it is done so and the *guṇa* (x) is obtained. This multiplier or *guṇa* will be the same for the original equation, but not the *labdhi.*

भाज्यतक्षणलाभो गुणेन गुणनीयः पश्चात् क्षेपतक्षणलाभेन संस्कार्यः । संस्कृतेन तेन गणितागता लब्धिः संस्कार्या सा लब्धिर्भवतीति ।(*BP.* p.110)– To find *labdhi*, multiply the *guṇa* by the quotient Q_1 of *bhājya* divided by divisor, add to it the quotient Q_2 of *kṣepa* divided by divisor and finally add the *labdhi* obtained by the first *kuṭṭaka.*

Note: This method is given by Kṛṣṇa.

Example : (*BG.* v. 72): Solve $5x + 23 = 3y$

After reduction, by using the rule quoted in **3.3.3.** the equation is $(5 - 3)x + (23 - 21) = 3y$, i.e. $2x + 2 = 3y$; dividing both 5 and 23 by the divisor 3 we have the remainders $R_1 = 2$. $R_2 = 2$ and quotients $Q_1 = 1$, $Q_2 = 7$ respectively.

According to Kṛṣṇa's rule (see **3.5.1.Case 2a below**) $x = y = 2$ since $(3 - 2 = 1)$

i.e. $$2(2) + 2 = 3(2) \quad ①$$

i.e. *guṇa* $x =$ *labdhi* $y = 2$

For $5x + 23 = 3y$, x will be same 2, y has to be determined. Now $5x + 23$ can be written as multiples of 3 plus a remainder

i.e. $$(3.1 + 2)x + (7.3 + 2)$$

since $x = 2$, substituting in above,

$$(3.1 + 2)2 + 7.3 + 2$$

$$= 3.1.2 + 7.3 + 2.2 + 2$$

$$= 3.1.2 + 7.3 + 3.2 \quad \text{from ①}$$

$$= 3(2 + 7 + 2) = 3(11) = 3y$$

Thus we obtain $y = 11$

3.3.4. General solution :

Example : Solve $ax + c = by$

a and c can be written as multiples of b plus a remainder.

i.e.
$$a = bQ_1 + R_1$$
$$c = bQ_2 + R_2$$

Let the new equation be

$$R_1x + R_2 = by$$

Let x_1, y_1 be a solution set for above equation

Then $$R_1x_1 + R_2 = by_1 \quad ①$$

Consider $ax + c = by$

Substituting for a and c

$$(bQ_1 + R_1)x + (bQ_2 + R_2) = by$$

According to *sūtra*, x_1 is a solution of the above ; but y has to be determined.

Now
$$\begin{aligned}
& (bQ_1 + R_1)x_1 + (bQ_2 + R_2) \\
&= bQ_1x_1 + R_1x_1 + bQ_2 + R_2 \\
&= bQ_1x_1 + bQ_2 + R_1x_1 + R_2 \\
&= bQ_1x_1 + bQ_2 + by_1 \qquad \text{from ①} \\
&= b(Q_1x_1 + Q_2 + y_1)
\end{aligned}$$

$\therefore$ The *labdhi* for the equation $ax + c = by$ is $(Q_1x_1 + Q_2 + y_1)$

3.4. European Method for Solving Diophantine Equation $ax \pm c = by$:

In the European method, the equation $ax \pm 1 = by$ is solved initially and then extended to $ax \pm c = by$.

By successive division $\frac{a}{b}$ is represented by a continued fraction (c.f), and the convergents of the c.f. are found.

$\frac{a}{b} = a_1 + \frac{1}{a_2 +}\ \frac{1}{a_3 +}\ \ldots\ldots\ \frac{1}{a_{n-1} +}\ \frac{1}{a_n}$ where the rational numbers a_1, a_2 .... a_n are the successive quotients of $\frac{a}{b}$

It is known[3] that any rational number is expressible as a (finite) simple continued fraction where $\frac{p_1}{q_1} = a_1$; $\frac{p_2}{q_2} = a_1 + \frac{1}{a_2} = \frac{a_1a_2 + 1}{a_2}$; ..... $\frac{p_n}{q_n} = \frac{a}{b}$

3. Refer C.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford, 1954, p. 135, Theorem 161.

are called the k^{th} convergents, $k = 1, 2, \ldots\ldots n$. It can be shown that $p_k q_{k-1} - q_k p_{k-1} = \pm 1$ according as k is even or odd.

In the given example $17x + 5 = 15y$ (of **3.3.**)

$$\frac{17}{15} = 1 + \frac{2}{15} = 1 + \cfrac{1}{\cfrac{15}{2}} = 1 + \cfrac{1}{7 + \cfrac{1}{2}} = 1 + \cfrac{1}{7 + \cfrac{1}{\cfrac{2}{1}}} = 1 + \cfrac{1}{7 + \cfrac{1}{2}}$$

Therefore

$$\frac{p_1}{q_1} = \frac{1}{1}\ ;\ \frac{p_2}{q_2} = 1 + \frac{1}{7} = \frac{8}{7}\ ;\ \frac{p_n}{q_n} = \frac{p_3}{q_3} = 1 + \cfrac{1}{\cfrac{15}{2}} = 1 + \frac{2}{15} = \frac{17}{15}$$

For $\qquad k = 3,\ p_3 q_2 - p_2 q_3 = -1$

i.e. $\qquad 17(7) - 15(8) = -1$

Thus $x = 7,\ y = 8$ is the solution for

$$17x + 1 = 15y$$

This result is used to solve $17x + 5 = 15y$

$$17x - 5(-1) = 15y$$

$$17x - 5(17.7 - 15.8) = 15y$$

$$17(x - 35) = 15(y - 40)$$

since $y - 40$ should be divided by 17 without remainder

$$y = 17t + 40$$

similarly, $\qquad x = 15t + 35$

where $\qquad t = 0,\ \pm 1,\ \pm 2 \ldots$

When $t = 0$, $y = 40$, $x = 35$

$t = -2$, $y = 6$, $x = 5$ and so on.

Thus from the solution of the equation of the form $ax \pm 1 = by$ is derived the solution of the equation $ax \pm c = by$.

Note: This European method appears already in the *Br.Sp.* of Brahmagupta (Ch. XVIII). Also refer to **3.7**.

3.5. Kṛṣṇa's analysis of the *kuṭṭaka* process :

Bhāskara has dealt with the *kuṭṭaka* in detail. Kṛṣṇa also explains every *sūtra* of Bhāskara elaborately. In this section, the methodology of *kuṭṭaka* process, is dealt with exhaustively. Initially the dividend *bhājya*, the divisor *hara* and additive *kṣepa* are kept positive. Examples, when one or more of the above is negative, are considered in later sub-sections.

3.5.1. *Kṣepa vicāra*:

Kṛṣṇa discusses the various situations with different values of *kṣepa*, starting with zero, elaborately. The discussion is as follows :

Let the general equation be $ax + c = by$, where x is *guṇa*, y is *labdhi* and a, b, c are arbitary but given numbers, of which c is *kṣepa* or additive.

Case 1 : (*BP*. p.90): क्षेप - अभावे When *kṣepa* additive is 0, the equation is $ax + 0 = by$, $a, b \neq 0$. There are two solutions to the above :

1) $ax + 0 = by \quad \Rightarrow \quad x = y = 0$

2) $a(b) + 0 = b(a) \quad \Rightarrow \quad x = b$ and $y = a$

Note : For any k, a positive integer, $x = kb$, $y = ka$ are solutions. But $x = b$, $y = a$ is the pair of least positive solutions. Also if k is a negative integer, $x = kb$, $y = ka$ are the solutions.

Case 2 : (*BP*. p. 93): यदि रूपं शेषं स्यात् : When remainder is 1 ; in this case we consider first $ax + c = by$ where $a \dot{-} b = 1$. The equation becomes

$$(b+1)x + c = by$$

We note that $x = y = -c$ is a solution, since

$$(b+1)(-c) + c = -bc$$

If c is positive we have thus solutions which are negative. By an earlier rule (*BG*. v. 62) : योगजे तक्षणाच्छुद्धे गुणाप्ती स्तो वियोगजे । धनभाज्योद्भवे तद्वद्भवेतामृणभाज्यजे ॥

$$x = tb - c,\ y = t(b+1) - c$$

where $t = 1, 2, \ldots$ is chosen so that x, y have positive values. To illustrate,

$$16x + 2 = 15y$$

has $x = y = -2$ as solution from which we obtain, $x = 15 - 2 = 13$, $y = 16 - 2 = 14$ as positive solutions. Thus, Case 2 is one in which, the solutions could be written out without recourse to any special process.

Case 2a: (*BP*. pp.100-01) : भाज्यशेषेणक्षेपेनिश्शेषभक्ते । The equation is $ax + c = by$, where a divided by b does not leave R remainder equal to 1 but remainder divides *kṣepa* without remainder.

Example : Solve $21x + 16 = 17y$

In this example, 21 divided by 17 leaves remainder $R = 4$ which divides *kṣepa* 16.

Then according to Kṛṣṇa, x (*guṇa*) = y (*labdhi*) = 4, not for the given equation but for $21x - 16 = 17y$. For,

$$21(4) - 4(4) = 17(4)$$

As in Case 2, using the *sūtra, yogaje takṣaṇāt śuddhe....*

$x = 17 - 4 = 13$, $y = 21 - 4 = 17$ is the solution of

$$21x + 16 = 17y$$

In cases other than Cases 1 and 2 above, the general procedure is detailed in Case 3 below :

Case 3 : (*BP.* p. 93): यदि रूपं शेषं न स्यात् – if the remainder is not 1

$ax + c = by$ where $a - b \neq 1$. Here a divided by b does not give remainder 1. This is illustrated by

Example : Solve $17x + 6 = 15y$

Then *kuṭṭaka* can be done as in Steps 3 to 6 of **3.3.**

apavartana	*Vallī*	
15) 17 (1	1	48
15	7	42
2) 15 (7	6	
14	0	
1		

Therefore

$$y\,(labdhi) = 48 - (17 \times 2) = 14$$

$$x\,(guṇa) = 42 - (15 \times 2) = 12$$

$$17(12) + 6 = 15 \times 14$$

Therefore when the *bhājya* divided by the *hāra* does not leave a remainder equal to 1, the normal *kuṭṭaka* method is followed.

Case 4 : (*BP.* p. 101) : भाज्यशेषेण भक्तः क्षेपः न शुद्ध्यति ।

In the equation : $ax + c = by$, a divided by b does not give a remainder 1; remainder also does not divide *kṣepa* without remainder.

Example : Solve $21x + 15 = 17y$

Here 21 divided by 17 gives remainder 4, but 4 does not divide 15 without remainder. The rule then is to follow the usual *kuṭṭaka* method.

```
17) 21 (1          1     75
    17             4     60
    --
     4) 17 (4     15
        16         0
        --
         1
```

x (*guṇa*) $= 60 - (17 \times 3) = 9$

y (*labdhi*) $= 75 - (21 \times 3) = 12$

i.e. $21(9) + 15 = 17 \times 12$

Case 5 : (*BP.* p. 101) : द्वितीयशेषेण क्षेपः शुद्ध्यति ।

Equation : $ax + c = by$

a divided by b does not give remainder = 1 ; remainder does not divide *kṣepa* without remainder initially ; but the second or later remainder in the division divides *kṣepa* without remainder.

Example : Solve $21x + 15 = 13y$

a divided by b does not give remainder = 1. Remainder 8 does not divide the *kṣepa* 15. Therefore proceed with *apavartana* till the remainder divides the *kṣepa* fully.

13) 21 (1	1
13	1
8) 13 (1	1
8	1
5) 8 (1	1
5	15
3) 5 (1	0
3	
2) 3 (1	
2	
1	

For easy calculation, Kṛṣṇa suggests, that the division may be stopped at the point where the remainder is 5, since 5 divides *kṣepa* 15 fully. $\frac{15}{5} = 3$ is taken as new *kṣepa*. Now the *Vallī* is :

	Vallī	
13) 21 (1	1	6
13	1	3
8) 13 (1	3	
8	0	
5		

We get the solution x (*guṇa*) = 3 and y (*labdhi*) = 6.

At the end of the discussion, Kṛṣṇa also expresses his opinion (*BP.* p. 102) : एवमस्मत्पक्षेऽस्ति लाघवम् । तदेवमपवर्तावश्यकत्वे गौरवमेवेति प्रतिभाति । – that his methods may be easier since they involve less number of steps and avoid long division. Thus Kṛṣṇa's methods are comprehensive,[4] simple and obtain least integer solutions in few steps.

4. *Cf.* Venkatesha Murthy, M.S. Rangachari and S. Baskaran, "Hindu Work on Linear Diophantine Equations", *Journal of Madras University*, Sec. B (1985), 48(1), pp. 1-19.

3.5.2. *Ṛṇabhājaka vicāra* (Negative Divisor discussed) :

In this section Kṛṣṇa takes for elaborate discussion, the case of negative divisors. Kṛṣṇa says (*BP.* p. 119) : हरमात्रस्य ऋणत्वेऽपि एतावेव लब्धिगुणौ किंतु लब्धिमात्रं ऋणं भागहारेऽपि चैवं निरुक्तम् इत्युक्तत्वात् . . . । – that when the divisor alone is negative, the *labdhi* also will be negative. This is according to the rule of division (*BG.* v. 11) : भागहारेऽपि चैवं निरुक्तम् । – which means positive divided by positive is positive, negative divided by negative is positive and positive divided by negative and vice versa, is negative.

In the example, $18x + 10 = -11y$, if all the three, i.e. the *bhājya* (18), *hāra* (11) and *kṣepa* (10) are kept positive, the solutions for x and y would be $x = 8$, $y = 14$, i.e. $18(8) + 10 = 11(14)$.

Then according to Bhāskara's rule, भागहारेऽपि चैवं निरुक्तम् । the solutions for the equation $18x + 10 = -11y$ would be as follows : New x would be the same 8, but new y would be -14.

Note : If x_1, y_1 is a solution of $ax + c = by$, then x_1, $-y_1$ is a solution of $ax + c = -by$ and vice versa.

In this context Kṛṣṇa notices the fact that if the rule is misinterpreted it would lead to wrong solutions (*BP.* p. 118) : केचित्[5] ऋणभाज्योद्भवे तद्वद्भवेतामृणभाजक इति पाठं कल्पयित्वा भाजकर्णत्वेपि स्वतक्षणाच्छोधनं कुर्वन्ति तदसदिति प्रतिभाति । – Some, taking it to mean "those deduced from a negative dividend being treated in the same manner, become solutions for a negative divisor" do a further reduction which is not warranted.

To illustrate this, Kṛṣṇa gives counter examples.

Case 1 : In the example treated earlier where the divisor alone is negative, if only absolute values of the solution is taken to apply the rule

5. The source referred to here by the word '*kecit*' is not traceable.

then, since $18\ (3) + 10 \neq (-11) \times 4$, $x = 11 - 8 = 3$, $y = 18 - 14 = 4$ are not solutions.

Case 2 : Again the rule (*BG.* v. 7) : संशोध्यमानं स्वमृणत्वमेति स्वत्वं क्षयस्तद्युतिरुक्तवच्च । may again be wrongly applied by taking the absolute value of the *guṇa.* In the above example,

$$x = -11 - 8 = -19, y = 18 - 14 = 4$$

is not a solution : $18\ (-19) + 10 \neq (-11) \times 4$

Therefore Kṛṣṇa. concludes that for negative divisor, Bhāskara's rule (*BG.* v. 11) : भागहारेऽपि चैवं निरुक्तम्। should be strictly followed (*BP.* p. 115) : अत्र भाज्यभाजकयोः विजातीययोः भागहारेऽपि चैवं निरुक्तमित्युक्तत्वात् लब्धं ऋणत्वं ज्ञेयम् । – Since *bhājya* and *hāra* have different signs, one divided by the other gives a quotient which is negative ; therefore *labdhi* will be negative.

3.5.3. *Ṛṇabhājya vicāra* (Discussion of the case of Negative Dividend) :

The rule of Bhāskara (*BG.* v. 62cd) says that :

धनभाज्योद्भवे तद्वद्भवेतामृणभाज्यजे ॥

– in the same way, the values for x and y when a is positive must be subtracted from b and a when a is negative.

This rule is now taken for elaborate discussion by Kṛṣṇa (*BP.* pp. 115-16), illustrated by the example : $-60x \pm 3 = 13y$

The problem should be first attempted keeping all the three *bhājya, hāra,* and *kṣepa* positive, i.e. $60x + 3 = 13y$

Vallī

```
13) 60 (4
    52
    ---
     8) 13 (1
         8
        ---
         5) 8 (1
            5
           ---
            3) 5 (1
               3
              ---
               2) 3 (1
                  2
                 ---
                  1
                 ---
```

4	69
1	15
1	9
1	6
1	3
3	
0	

$y' = 69 - 60 = 9$

$x' = 15 - 13 = 2$

Since the number of quotients is 5 (odd), applying the rule (*BG.* v. 60) :

एवं तदेवात्र यदा समस्ताः स्युर्लब्धयश्चेद्विषमास्तदानीम् ।
यथागतौ लब्धिगुणौ विशोध्यौ स्वतक्षणाच्छेषमितो तु तौ स्तः ॥

– values of *labdhi* and *guṇa* obtained must be subtracted from their respective *takṣaṇas* to get the correct values of the *labdhi* and *guṇa*. The actual solutions of the equation $60x + 3 = 13y$ are $y = 60 - y' = 51$; $x = 13 - x' = 11$.

Note: *Takṣaṇa* has two meanings:

(i) Modulo operation, i.e., a division for obtaining the remainder

(ii) The divisor in that modulo operation.

Case 1 : Solve $-60x + 3 = 13y$

The *guṇa* and *labdhi* for positive *bhājya* (+60) and positive *kṣepa* (+3) are 11 and 51 respectively. These after subtraction of multiples of *bhājaka* and *bhājya* from 13, 60 become 2, 9 ; but 2, –9 are solutions for negative *bhājya* (*viz.* –60) and positive *kṣepa*, according to the rule (*BP.* p. 116) :

भाज्यभाजकयोः मध्ये एकस्यैव ऋणत्वे लब्धि मात्रस्य ऋणत्वं ज्ञेयम् ।, i.e. the *labdhi* obtained for $60x + 3 = 13y$ is to be taken with a negative sign.

It is also clear that for $-60x - 3 = 13y$ the solutions are $11, -51$, where the dividend and *kṣepa* of $60x + 3 = 13y$ are taken with a negative sign, as in

Case 2 : Solve $-60x - 3 = 13y$

For, in this case, $60x + 3 = -13y$

This can also be treated as a case of negative divisor.

Therefore solutions for cases where *bhājya, hāra* and *kṣepa*, are not positive can be derived from the solutions of the corresponding equations with their absolute value in place.

3.6. *Pūrveṣām kuṭṭaka-vyabhicāra vicāra* (Discussion on the errors of earlier Authors) :

Bhāskara seems to have noticed that by the rule of some earlier writer, errors would arise in case dividend is negative (*BG.* p. 29) : "एकस्मिन् ऋणगते गुणाप्ती "द्वौ राशी क्षिपेत् तत्र–" इत्यादिना परोक्तसूत्रेण लब्धौ व्यभिचारः स्यात् । Kṛṣṇa discusses this in detail and explains the situation where these errors will occur.

The problem arises when the *bhājya* and *kṣepa* are of opposite signs. Bhāskara wonders (*BG.* p.29) : भाज्ये भाजके वा ऋणगते परस्परभजनात् लब्धय ऋणगताः स्थाप्या इति किं तेन प्रयासेन । "If the divisor or dividend is negative, the quotients obtained from mutual division, which are negative, should be placed in the *vallī*. Why should there be such an effort when the rule is clear?". Kṛṣṇa also adds "when the rule on the negative dividend is clear why has so much effort been wasted by others."

Bhāskara has clearly stated the rules when *bhājya, hāra,* and *kṣepa* are negative. In such a case, Kṛṣṇa wonders why people should try otherwise (*BP.* p. 116) : एवम् ऋणभाज्ये अपि अप्रयासेनैव कुट्टकसिद्धौ सत्यामपि अन्यैः वृथा प्रयासः कृतः ।

Kṛṣṇa shows how this leads to unnecessary difficulty (*BP.* p. 116) : अत्र क्षेपस्य ऋणत्वे धनत्वे वा उपान्तिमेन स्वोर्ध्वे हत इत्यादिकरणे धन ऋणत्व अवधानेन प्रयासगौरवं द्रष्टव्यम् । न केवलं प्रयासोऽपि तु लब्धौ व्यभिचारोऽपि । – by changing the sign of the additive, the method becomes harder. He adds that not only it is difficult to solve, but the quotient *labdhi* will be wrong. He also says (*BP.* p. 117) : नह्यत्र लब्धौ एव इति अवधारणमस्ति किंतु लब्धौ इति उपलक्षणं तेन गुणोऽपि व्यभिचार: स्यादित्यर्थ: । – Though the word *labdhi* is used, it also includes *guṇa.*

Kṛṣṇa deals with two situations : i) when quotients are odd ; ii) quotients are even.

3.6.1. Odd quotients :

Case 1 : भाज्यभाजकक्षेपाणां धनत्वे लब्धीनां विषमत्वे – When *bhājya, hāra* and *kṣepa* are positive and number of quotients is odd.

Example : Solve $60x + 3 = 13y$

Vallī

4	69	After reduction
1	15	$y = 69 - 60 = 9$
1	9	$x = 15 - 13 = 2$
1	6	
1	3	
3		
0		

Without reduction for odd number of quotients as per *sūtra BG.* v. 60, (refer **3.5.3**) the top entries in the *vallī* do not give solutions ; for,

$$60(2) + 3 \neq 13(9)$$

On the other hand, if reduction is done as necessary, then

$$y = 13 - 2 = 11\,; \qquad x = 60 - 9 = 51$$

and $$60(11) + 3 = 13\,(51)$$

So when all of *bhājya, hāra* and *kṣepa* are positive and the quotients are odd, the second reduction is essential.

Case 2 : When *bhājya* is negative and there are odd number of quotients.

Solve $$-60x + 3 = 13y$$

Vallī

−4	−69	After reduction
−1	+15	$y = 69 - 60 = 9$
−1	−9	$x = 15 - 13 = 2$
−1	+6	
−1	−3	
3		
0		

The solutions obtained when all three, *bhājya, hāra* and *kṣepa* are positive are (2,9). Remembering the rule (*BG.* v. 62cd) : धनभाज्योद्भवे तद्वद्भवेतामृणभाज्यजे । keeping the solutions as (2,9) only the *labdhi* is made negative, since the *bhājya* and *hāra* are of different signs.

i.e. $$-60\,(2) + 3 = 13\,(-9)$$

Instead, if reduction is done as for odd number of quotients, we do not get the solution.

For, $$-60\,(11) + 3 \neq 13\,(51)$$

So, for negative *bhājya,* the second reduction is not necessary.

3.6.2. Even quotients :

If there are even number of quotients as for example in the case of the equation $18x + 10 = -11y$.

Vallī	
−1	50
−1	−30
−1	20
−1	−10
10	
0	

After reduction the solutions are 14, −8

But $18(-8) + 10 \neq -11(14)$

On the other hand, a further reduction would mean that

$y = (18 - 14) = 4$, $x = -(11 - 8) = -3$

(−3, 4) would be a solution. So for even quotients the second reduction is necessary if divisor is negative.

3.6.3. Conclusions drawn by Kṛṣṇa :

a) The problem should be solved keeping all the three, *bhājya, hāra* and *kṣepa* positive.

b) The necessary reduction should be done to obtain the quotients, *labdhi* and *guṇa.*

c) Further reduction should be done for odd number of quotients.

d) After the *labdhi* and *guṇa* are obtained keeping all three positive, those for negative *kṣepa, bhājya* and for *hāra* should be derived.

It is seen that error (*vyabhicāra*) occurs in two situations (*BP.* p. 118) : अत्र समलब्धिषु हरस्य ऋणत्वे सति विषमलब्धिषु भाज्यस्य ऋणत्वे सति वा पूर्वेषां कुट्टके व्यभिचार इति निष्कर्षः । – (a) When the *bhājya* is negative and reduction is done for odd quotients ; (b) when the *hāra* is negative and reduction is not done for even quotients.

Remarks : The actual words used by Bhāskara in this connection in his explanation (*BG.* p. 29) : . . . तथा कृते सति भाज्यभाजकयोः एकस्मिन् ऋणगते गुणाप्ती "द्वौ राशी क्षिपेत् तत्र" इत्यादिना परोक्तसूत्रेण लब्धौ व्याभिचारः स्यात् are not quoted verbatim by Kṛṣṇa. We cannot come to a conclusion as to who is referred to by Bhāskara when he says *parokta sūtreṇa.* Kṛṣṇa (*BP.* p. 116) uses the phrases *pūrveṣām kuṭṭake* and *anyaiḥ vṛthā prayāsaḥ kṛtaḥ.*

3.7. *Sthirakuṭṭaka* :

The simple indeterminate equation $ax \pm 1 = by$ is solved in exactly the same way as the equation $ax \pm c = by$ and is only a special case of the latter. On account of its special use in astronomical calculations, it has received separate consideration at the hands of most Indian algebraists. The above equation has been generally called by the name of *Sthirakuṭṭaka* (constant pulverizer). Pṛthūdakasvāmin in his commentary on Brahmagupta's *Br.Sp.* sometimes uses the word *dṛḍhakuṭṭaka.*[6]

Indian mathematicians were aware that the equations of the form $ax \pm 1 = by$ could be solved by the *kuṭṭaka* method. They also knew that from the solutions of the above equations, solutions for $ax \pm c = by$ could be deduced.

Sthirakuṭṭaka or equations of the form $ax \pm 1 = by$ are of great importance says Kṛṣṇa (*BP.* p. 122) : अस्ति अत्र ग्रहगणिते स्थिरकुट्टकस्य महत्प्रयोजनम् । – they are of use in calculating the position of the planets at a particular time. In any example where *graha-ahargaṇa* (position of the planet) and the elapsed days are to be determined, Kṛṣṇa says (*BP.* p. 102): तत्र ऋणक्षेपस्य विकलाद्यग्रस्य अनियतत्वात् प्रतिप्रश्नात् ततस्ततो विकलाद्यग्रात् कुट्टककरणे अस्ति भूयान् प्रयासः । – that the dividend is sixty which is fixed (being the number of seconds in a minute) as also the *kudināni* or days in one *kalpa* which is given. The *vikalāśeṣa* (remaining seconds) is a negative additive and not

6. Datta and Singh, *op.cit.*, Vol. II, p.117.

a fixed quantity. In solving such problems, the method of *Sthirakuṭṭaka* is applied in order to avoid (*BP* p.124) : दीर्घवल्लीसंभूतयोः लब्धिगुणयोः साधने अस्ति गौरवम् । – the long *vallī* which would otherwise arise if the days elapsed is a big number.

Kṛṣṇa adds that this could be easily solved in the following manner (*BP* p.124) : स्थिरकुट्टके तु रूपमृणं प्रकल्प्य लब्धिगुणौ स्थिरौ कृत्वा तत्र विकलाशेषेण तयोर्गुणने सति स्वस्वहारेण तक्षणे च सति स्वाभीप्सित लब्धिगुणसिद्धिः इति अतिलाघवमस्ति । – When the additive is assumed to be negative or positive number 1, the method becomes simple. After obtaining the *labdhi* and *guṇa*, they can be multiplied by the actual *vikalāśeṣa* to obtain the true *labdhi* and *guṇa*.

3.7.1. The *Sthirakuṭṭaka* Method :

Example :

Let the equation be $17x + 1 = 15y$

By *kuṭṭaka* method $x = 7$, $y = 8$ is a solution ; multipling both x and y by 5 (say) we get the solutions for the equation

$$17x + 5 = 15y$$

i.e.

$$17\,(35) + 5 = 15\,(40)$$

By reduction (*takṣaṇa*) we also have $x = 35 - 30 = 5$, $y = 40 - 34 = 6$ as solutions of $17x + 5 = 15y$

Similarly, for the example $17x - 1 = 15y$.

$x = 8$, $y = 9$ is a solution set ;

For the equation $17x - 5 = 15y$, multiplying both x and y by 5 we get the solutions (40, 45). $17\,(40) - 5 = 15\,(45)$ will be a solution set.

By reduction we also have $x = 40 - 30 = 10$, $y = 45 - 34 = 11$ as solutions for $17x - 5 = 15y$.

3.7.2. Computation of planetary positions :

While dealing with the computation of planetary position, Kṛṣṇa takes the following rule from *Siddhānta Śiromaṇi.* (Graha-ānayana adhyāya, v.4), which is to be kept in mind for the calculation.

द्युचरचक्रहतो दिनसंचय: कहहृतो भगणादिफलं ग्रह: । – The *ahargaṇa* multiplied by the number of sidereal revolutions of a planet and divided by the number of civil days in a *kalpa* gives the (position) of the planet i.e. its number of revolutions upto the day concerned, both integral and fractional.

For the sake of knowing the remainder of seconds i.e. *vikalāśeṣa*, the position of the planet is obtained by the above rule. Thus in an example, the revolutions of planet are imagined to be 9, civil days (*kudināni*) 19 and elapsed days (*ahargaṇa*) 13.

"Applying the rule of three (*trairāśika*, *viz.*, ratio and proportion), if in C the number of civil days in a *kalpa*, the planet makes P sidereal revolutions, how many revolutions would have been made in A, the *ahargaṇa*? The answer is : $\frac{A \times P}{C}$. In this, the integral quotient gives the number of complete revolutions made ; the remainder multiplied by 12 and divided by C again, gives the number of *rāśis* covered by the planet from the zero point of the zodiac, and again the remainder multiplied by 30 and divided by C gives the number of degrees covered in the next *rāśi* ; proceeding thus, the planetary position could be had next in minutes and then in seconds of arc." [7]

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7. *Siddhānta Śiromaṇi* of Bhāskaracārya, ed. and tr. by D. Arka Somayaji, Rashtriya Sanskrit Vidyapeeta, Tirupati, 2000, pp. 32-3.

Therefore, by the rule of *trairāśika* $\left(\frac{9\times13}{19}\right)$, the *bhagaṇādigraha* is expressed as 6 (*gatabhagaṇa*) completed revolutions of the planet, 1 zodiac sign (*rāśi*), 26 degrees (*amśa*), 50 minutes (*kalā*), 31 seconds (*vikalā*), and remainder of seconds 11 (*vikalāśeṣa*), i.e. *bhāgaṇādigraha* is expressed as : 6 / 1 / 26 / 50 / 31.

''Now, by inversion, to find the planet's place from remainder of seconds, if the remainder of seconds be deducted from the remainder of minutes multiplied by sixty, then the difference divided by terresterial days will yield no residue . . . and the quotient will be seconds. Now in the problem, sixty and the remainder of seconds (as also the terresterial days in a *kalpa*) are known ; and thence to find the remainder of minutes, a multiplier is to be sought, such that sixty being multiplied by it, and the subtractive quantity (remainder of seconds) being taken from the product, the difference may be divisible by terresterial days without residue ; and this precisely is matter for investigation of (*kuṭṭaka*) the pulverizing multiplier''.[8]

3.7.3 Example given by Bhāskara (*BG.* v. 75) and explained by Kṛṣṇa :

कल्प्याऽथ शुद्धिर्विकलावशेषं षष्टिश्च भाज्यः कुदिनानि हारः ।
तज्जं फलं स्युर्विकला गुणस्तु लिप्ताग्रमस्माच्च कलालवाग्रम् ।।

Let remainder of seconds be the negative *kṣepa*, sixty the dividend, and *kudināni* the divisor. The quotient thus arrived will be the seconds ; multiplier will be the remainder of minutes. From this, the minutes and remainder of degrees and so on are found (by working upwards).

For the above example Kṛṣṇa shows the method of working backwards in finding the *graha* and *ahargaṇa*. *Bhājya* = 60 ; *kudināni* = 19; *vikalāśeṣa* = 11, and the equation is $60x - 11 = 19y$. By *kuṭṭaka*, the least positive solution set is (10,31)

8. H.T. Colebrooke, *op.cit.*, p. 167

i.e. *labdhi* = 31 *vikalā* (seconds), *guṇa* = 10 *kalāśeṣa* (remainder of minutes)

Kuṭṭaka is done again with 10 as *kalāśeṣa, viz.* $60x - 10 = 19y$. By *kuṭṭaka*, the least positive solution set is (16,50). So, *labdhi* = 50 *kalā* (minutes), *guṇa* = 16 *bhāgaśeṣa* (remainder of degrees)

Kuṭṭaka is again performed with 16 as *bhāgaśeṣa* $30x - 16 = 19y$. By *kuṭṭaka*, the least positive solution set is (17,26), i.e. *labdhi* = 26 *bhāga* (degrees), *guṇa* = 17 *rāśiśeṣa* (remainder of zodiac signs).

Kuṭṭaka is performed again with 17 as *rāśiśeṣa* $12x - 17 = 19y$. By *kūṭṭaka*, the least positive solution set is (3,1), i.e. *labdhi* = 1 *rāśī* (zodiac sign), *guṇa* = 3 *bhagaṇaśeṣa* (remainder of revolutions).

Kuṭṭaka is again performed with 3 as *bhagaṇaśeṣa* $9x - 3 = 19y$. By *kūṭṭaka*, the least positive solution set is (13,6), i.e. *labdhi* = 6 *gatabhagaṇa* (number of revolutions elapsed), *guṇa* = 13 *ahargaṇa* (elapsed days).

Note : It is surprising that neither Bhāskara nor the commentators Kṛṣṇa or Sūryadāsa use the *Sthirakuṭṭaka* to solve the example. Most probably there is no significant advantage in using *Sthirakuṭṭaka*, since small numbers are involved in this particular illustration.

3.7.4. Kṛṣṇa's observations :

1. The reason for the use of *kuṭṭaka* here :

The planetary position is known by working backwards from *vikalāśeṣa* (*BP.* p. 125) : तस्मात् षष्टिः कलाशेषेण गुणिता विकलशेषेण ऊना कुदिनभक्ता निःशेषा स्यात् लब्धिस्तु विकलाः स्युः । प्रकृते षष्टिः विकलाशेषं च ज्ञायते । केवलं कलाशेषं न ज्ञायते । तत् ज्ञापनार्थम् उपायः षष्टिः येन गुणिता सती विकलाशेषेण ऊना कुदिनभक्ता निःशेषा भवेत् तदेव कलाशेषं स्यात् । अयमर्थश्च कुट्टकस्य विषयः षष्टिः केन गुणिता विकलाशेषेण रहिता कुदिनभक्ता निःशेषा स्यादिति प्रश्ने पर्यवसानात् । – Since 60 seconds is equal to one minute, 60

multiplied by *kalāśeṣa* (remainder of minutes) and divided by *kudināni* will give *vikalā* (seconds) with or without remainder. Usually, *vikalā* (*bhājya*) sixty and remainder of seconds *vikalāśeṣa*, are known. To get the unknown *kalāśeṣa, kuṭṭaka* is employed.

2. The rule (*BG.* v. 67) : इष्टाहतस्वस्वहरेण युक्ते ते वा भवेतां बहुधा गुणाप्ती । is not valid here, says Kṛṣṇa (*BP.* p. 127) : अत्रेदमवधेयम् । विकलाशेषात् ग्रहानयने विकलाशेषम् ऋणक्षेपः षष्टिः भाज्या कल्पकुदिनानि हारः इति प्रकल्प्य कुट्टकेन यो लब्धिगुणौ तौ इष्टाहतस्वस्वहरेण युक्तौ न विधेयौ । योजने हि षष्टितोऽधिका स्यात् गुणश्च कुदिनतोऽधिकः स्यात् । न चैतत् संभवति यतो लब्धिः विकलाः गुणश्च कलाशेषम् । न हि कलाः षष्टितोऽधिकाः संभवन्ति न वा कलाशेषं कुदिनतोऽधिकं संभवति । – When *kuṭṭaka* is used with 60 as *bhājya* (dividend), *vikalāśeṣa* (remainder of seconds) as *kṣepa* (additive) and *kudinani* as divisor, the corollary '*iṣṭāhatasvasvahareṇa*' etc. is not valid.

Taking the first equation $60x - 11 = 19y$, according to the above *sūtra* : *labdhi* = $60t + 31$; *guṇa* = $19t + 10$; where t is any number 0, ±1, ±2, ±3 . . . Accordingly *labdhi* and *guṇa* can have infinite values. But here, in the above example, *vikalā* (seconds) cannot be more than 60, i.e. $60t + 31$ cannot be greater than 60 and *kalāśeṣa* (remainder of minutes) cannot be more than *kudināni* or civil days i.e. $19t + 10$ cannot be greater than 19. Therefore the above *sūtra* cannot be applied here.

3.8. *Saṃśliṣṭakuṭṭaka*[9] (Conjunct Pulverisor – Simultaneous Linear Diophantine Equation) :

The last topic dealt with in the *Kuṭṭaka* chapter of *BG* is *Saṃśliṣṭakuṭṭaka* or conjunct pulveriser. Here the dividend could be many, but the divisor is common. Bhāskara explains the procedure of *Saṃśliṣṭakuṭṭaka* as follows (*BG.* v. 76) :

9. *Cf.* Pushpa Kumari Jain, *The Sūryaprakāśa of Sūryadāsa*, Vol. I, Oriental Institute, Vadodara, 2001, pp. 236-39.

एको हरश्चेद्गुणकौ विभिन्नौ तदा गुणैक्यं परिकल्प्य भाज्यम् ।
अग्रैक्यमग्रं कृत उक्तवद्यः संश्लिष्टसंज्ञः स्फुटकुट्टकोऽसौ ॥

– If the divisor be the same, but the multipliers different, then making the sum of the multipliers, the dividend and the sum of remainders, the remainder (of the *kuṭṭaka*), the process of *kuṭṭaka* is carried out. This is called *Samśliṣṭakuṭṭaka.*

Kṛṣṇa explains the process thus (*BP.* p. 128) : तत्र गुणकैर्पृथक् गुणितो युक्तश्चेत् गुणकयोगेनैव गुणितः स्यात् । अतो गुणकयोग एवात्र गुणः शेषयोग एव शेषम् । – Now the quantitiy multiplied by the sum of the multiplicators is the same as if severally multiplied by the multiplicators and the products added together. Therefore the sum of the multiplicators is taken as the *guṇa* (and employed as *bhājya*) and sum of *kṣepa* is taken as remainder (and employed as *kṣepa*).

3.8.1. Example given by Bhāskara (*BG.* v. 77) :

कः पञ्चनिघ्नो विहृतस्त्रिषष्ट्या सप्तावशेषोऽथ स एव राशिः ।
दशाहतः स्याद्विहृतस्त्रिषष्ट्या चतुर्दशाग्रो वद राशिमेनम् ॥

– What is that number which multiplied by 5 and divided by 63 gives 7 as remainder and when multiplied by 10 and divided by 63 gives 14 as remainder?

Here the divisor is the same but the multipliers are different i.e. equations are of the form

$$63x + 7 = 5y$$

$$63x + 14 = 10y$$

Kuṭṭaka is used to solve the equation

$$63x + 21 = 15y$$

Dividing by 3, we have the equation

$$21x + 7 = 5y$$

Solutions sets are (3,14), (8,35) . . . Therefore the least value of y which satisfy the equation is 14.

General form of the equation :

$$ax_1 + c_1 = b_1y$$

$$ax_2 + c_2 = b_2y$$

$$ax_3 + c_3 = b_3y$$

Adding

$$a(x_1 + x_2 + x_3) + (c_1 + c_2 + c_3) = y\,(b_1 + b_2 + b_3)$$

Taking

$$x_1 + x_2 + x_3 = X\ ,\ b_1 + b_2 + b_3 = B\cdot,\ \ c_1 + c_2 + c_3 = C$$

we have the equation

$$aX + C = By$$

By *kuṭṭaka,* x can be obtained. Of the many values of x atleast one may/may not satisfy all the above equations.

3.8.2. Kṛṣṇa's own example (*BP.* p. 128) :

$$19x_1 + 1 = 2y$$

$$19x_2 + 11 = 3y$$

$$19x_3 + 2 = 4y$$

which in the 'added form' yields

$$19(x_1 + x_2 + x_3) + 14 = 9y$$

Where $x_1 + x_2 + x_3 = x$. Solving the last equation by *kuṭṭaka*, solution sets are (4,10) (13,29) . . . Therefore least value $y = 10$ satisfies all the equations.

Two astronomical examples are referred to by Kṛṣṇa (*Siddhānta Śiromaṇi,* Praśnādhyāya, vv.10, 13) where *Samśliṣṭakuṭṭaka* is employed.

To sum up, Bhāskara has dealt with exhaustively all types of the equation dealing with *kuṭṭaka.* These are explained in detail by Kṛṣṇa.

1) Where either of the *kṣepa* or *bhājya* is negative, reduction has to be done. Moreover when only the *bhājya* is negative, the quotient alone is made negative.

2) When both the *kṣepa* and *bhājya* are negative then $ax + c$ is negative, because, if x is assumed to be positive then both ax and c are negative. Then reduction is not required. But the quotient needs to be made negative (since $ax + c$ is negative, by is negative and therefore y is negative).

3) When the *hāra* or divisor alone is negative it is equivalent to the case above where the *kṣepa* and *bhājya* are negative ($-ax - c = by$ is the same as $ax + c = -by$). Here also the quotient alone is made negative.

4) The case where both *kṣepa* and *hāra* are negative is equivalent of the case where dividend alone is negative that is $ax - c = -by = (-ax) + c = by$.

5) The case where *bhājya* and *hāra* are negative is equivalent of the case where the *kṣepa* alone is negative $-ax + c = -by \Rightarrow ax - c = by$.

6) Finally when all three – *bhājya*, *hāra* and *kṣepa* are negative, it is the same as all three of them being positive $-ax - c = -by \Rightarrow ax + c = by$.

CHAPTER - 4

VARGA-PRAKṚTI — CAKRAVĀLA

(Indeterminate equation of the second degree - cyclic method)

We now trace the history of the *varga-prakṛti* equation and the *cakravāla* method as propounded by Bhāskara. Earlier to him, Brahmagupta had given the *bhāvanā* method to solve *varga-prakṛti* and Śrīpati, the auxiliary equations. Kṛṣṇa has given proofs for the *cakravāla* method and Śrīpati's rule. It is to be emphasized that equations of the form $Nx^2 + 1 = y^2$ were discussed by Indian algebraists much earlier than Europeans. Other forms of the *varga-prakṛti* equation are also discussed here.

4.1. *Varga-prakṛti* :

Indian algebraists were the first to evolve and describe algorithms for finding all integer solutions of linear Diophantine equations or what in Indian Mathematics is termed as *kuṭṭaka* discussed in the previous chapter. From the time of Brahmagupta, mathematicians in India were attempting the harder problem of solving equations of the second degree. As early as 628 A.D., Brahmagupta gave a partial solution to the problem of solving $Nx^2 + 1 = y^2$. Thereby, as Michael Atiyah, a leading mathematician of the present times, has aptly put it, he has "made important contributions to what is now known (incorrectly) as Pell's equation."[1] The fact that the equation has infinite solutions was also known to Brahmagupta is evident from his method of *bhāvanā*.

The motivation for solving such equation was probably to find rational approximation for surds. From $Nx^2 + 1 = y^2$ we find

$$\left|\sqrt{N} - \frac{y}{x}\right| \leq \frac{1}{2xy} \quad \text{so that} \quad \sqrt{N} \approx \frac{y}{x} \quad \text{if } x \text{ and } y \text{ are large.}$$

1. M. Atiyah, "Mathematics as a basic science", *Current Science*, Vol. 65, No. 12 (1993), p. 913.

For instance $2(408)^2 + 1 = 577^2$ so that $\sqrt{2} \approx \frac{577}{408}$

It is noteworthy that this is the value of $\sqrt{2}$ given in *Śulba sūtra*[2] where

$\sqrt{2}$ is given as $= 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} = \frac{577}{408}$

4. 1. 1. The term *Varga-prakṛti* :

In the very beginning of the chapter on *varga-prakṛti* , Kṛṣṇa traces the origin of the name *varga-prakṛti* (*BP.* p. 130) :

वर्गः प्रकृतिर्यत्रेति वर्गप्रकृतिः । यतोऽस्य गणितस्य यावदादिवर्गः प्रकृतिः । यद्वा यावदादिवर्गेषु प्रकृतिभूताद् अङ्काद् इदं गणितः प्रवर्तत इति वर्गप्रकृतिः । अत्र यावद्वर्गादिषु प्रकृतिभूतो यो अङ्कस्सः प्रकृतिशब्देनोच्यते स चाव्यक्तवर्गगुणक एव । अतो अत्र पद साधने वर्गस्य यो गुणः स प्रकृतिशब्देन व्यवह्रियते । - "That in which the *varga* (square) is the *prakṛti* (nature) is called the *varga-prakṛti* ; for the square of *yāvat* etc., is the *prakṛti* (origin) of this (branch of) mathematics. Or, because this (branch of) mathematics has originated from the number which is the *prakṛti* of the square of the *yāvat* etc., so it is called the *varga-prakṛti.* In this case the number which is the multiplier of the square of *yāvat*, etc., is denoted by the term *prakṛti.* (In other words) it is the coefficient of the square of the unknown"[3]. Therefore in this determination of the square root, the multiplier of the square is treated with the word *prakṛti.*

In the general equation $Nx^2 + k = y^2$, where k is positive, N is the *prakṛti*, x is called *hraśva* or *kaniṣṭha*, the lesser root and y is called the *jyeṣtha* or greater root. However the words 'lesser' and 'greater' are not always accurate. Suppose $x = a$ and $y = b$, is a solution of $Nx^2 + k = y^2$ where k is positive then $a < b$. Note that if k is negative, then $b < a$. In that case, it will be ambiguous to call 'a' the lesser root and 'b' the greater root. Therefore, Kṛṣṇa makes this just observation (*BP.* pp. 130-31) :

2. S.N. Sen and A.K. Bag, *The Śulbasūtras of Baudhāyana, Āpastamba, Kātyāyana and Mānava* with text and Eng. tr. and com., Indian National Science Academy, New Delhi, 1983, p. 161.
3. Datta and Singh, *op.cit.*, Vol. II, p.142

अन्वर्थाश्चैतास्संज्ञाः । यत्र तु क्षेपवियोगात्कुत्रचित् ज्येष्ठपदं ह्रस्वपदाद् अल्पं भवति तत्रापि भावनया ह्रस्वपदाद् अधिकमेव भवति । – These terms are significant. Where the greater root is sometimes smaller than the lesser root owing to the *kṣepa* being negative, there also it becomes greater than the lesser root after the application of the principle of *bhāvanā.*

4. 2. The Method of Brahmagupta's *Bhāvanā* as explained by Kṛṣṇa :

The most fundamental step in Brahmagupta's method for the general solution in positive integers of the equation, $Nx^2 + 1 = y^2$, where N is any non-square integer, is to consider two auxiliary equations $Nx^2 + k_i = y^2$, $i = 1, 2$ with k_i being chosen from $k_i = \pm 1, \pm 2, \pm 4$. A procedure known as *bhāvanā*, applied repeatedly, wherever necessary, helps us in deriving atleast one possible solution of the original *varga-prakṛti viz.,* $Nx^2 + 1 = y^2$. This solution, by means of the same principle of *bhāvanā*, yields an infinite number of solutions. Brahmagupta could find this auxiliary equation only by trial and error. Later improvement upon this with *cakravāla* is usually attributed to Bhāskara and traced back to Jayadeva.

Bhāvanā is of two kinds – *samāsa-bhāvanā* and *antara-bhāvanā.* Kṛṣṇa explains (*BP.* p.131) that if the products are added, it is as the name suggests, *samāsa-bhāvanā.* On the other hand, if one product is subtracted from the other, it is *antara-bhāvanā.* Kṛṣṇa says : तत्र पदयोः महत्वे अपेक्षिते समासभावनामाह । – If solutions of higher value are required *samāsa-bhāvanā* is employed. In the same way, – अथ पदयोः लघुत्वे अभीप्सितेतरभावनामाह । – when solutions of lesser value are required *antara-bhāvanā* is used.

Also, if the same roots are set down in the column and *bhāvanā* is performed, it is called *tulya-bhāvanā.*

4.2.1. Brahmagupta's *Bhāvanā* :

Let the general equation be $Nx^2 \pm k = y^2$. Let $(x = x_1,\ y = y_1,\ k = k_1)$ and $(x = x_2,\ y = y_2,\ k = k_2)$ be values satisfying the equation $Nx^2 + k = y^2$. Form the *paṅkti* (columns) as follows :

prakṛti	*kaniṣṭha*	*jyeṣṭha*	*kṣepa*
N	x_1	y_1	k_1
	x_2	y_2	k_2

Then, by *vajrābhyāsa* (cross multiplication), and direct multiplication, the new roots will be :

$$x = x_1y_2 + x_2y_1$$
$$y = Nx_1x_2 + y_1y_2$$
$$k = k_1k_2$$

In particular if $k_1 = k_2 = 1$ then the equations are

$$Nx_1^2 + 1 = y_1^2\ ,\quad Nx_2^2 + 1 = y_2^2$$

In case of the equation $Nx^2 + 1 = y^2$, *tulya-bhāvanā* yields $NX^2 + 1 = Y^2$ which is $N(2xy)^2 + 1 = (y^2 + Nx^2)^2$, where $X = 2xy$ and $Y = (y^2 + Nx^2)$

An infinite number of solutions can be obtained if one solution is known. It will be shown that *bhāvanā* further helps in solving $Nx^2 + 1 = y^2$ provided we know a solution for $k = -1,\ \pm 2,\ \pm 4$.

In terms of solutions (x_1y_1) for $Nx_1^2 + k = y_1^2$, where k is as in one of the five choices below, the solutions for $Nx^2 + 1 = y^2$ can be written out as indicated against each case through *tulya-bhāvanā* :

1. $k = -1$ $\quad y = y_1^2 + Nx_1^2$, $\quad x = 2x_1y_1$

2,3. $k = \pm 2$ $\quad y = \dfrac{y_1^2 + Nx_1^2}{2}$, $\quad x = x_1y_1$

4. $k = +4$ $\quad y = (y_1^2 + 2)\left[\frac{1}{2}(y_1^2 + 1)(y_1^2 + 3) - 1\right]$

$$x = \frac{x_1 y_1 (y_1^2 + 1)(y_1^2 + 3)}{2}$$

5. $k = -4$ $\quad y = \frac{y_1^2 - 2}{2}$, $\quad x = \frac{x_1 y_1}{2}$ when y_1 is even

$$y = \frac{y_1^2 - 3}{2}, \quad x = \frac{x_1(y_1^2 - 1)}{2} \text{ when } y_1 \text{ is odd.}$$

It would appear that Brahmagupta did not justify his procedure. Bhāskara took it up in his *BG*.

4.3. *Bhāvanā Upapattis* :

In the following, *upapattis* 1,2, and 4 are given by Kṛṣṇa.

4.3.1. *Upapatti* 1 :

Let the auxilary equations be

$$Nx_1^2 + k_1 = y_1^2 \quad ①$$

$$Nx_2^2 + k_2 = y_2^2 \quad ②$$

multiplying the first equation by y_2^2

$$Nx_1^2 y_2^2 + k_1 y_2^2 = y_1^2 y_2^2$$

$$Nx_1^2 y_2^2 + k_1(Nx_2^2 + k_2) = y_1^2 y_2^2 \quad (\text{since } Nx_2^2 + k_2 = y_2^2)$$

$$Nx_1^2 y_2^2 + k_1 Nx_2^2 + k_1 k_2 = y_1^2 y_2^2$$

$$Nx_1^2 y_2^2 + (y_1^2 - Nx_1^2)\, x_2^2 N + k_1 k_2 = y_1^2 y_2^2$$

$$Nx_1^2 y_2^2 + Nx_2^2 y_1^2 - N^2 x_1^2 x_2^2 + k_1 k_2 = y_1^2 y_2^2$$

$$Nx_1^2 y_2^2 + Nx_2^2 y_1^2 + k_1 k_2 = y_1^2 y_2^2 + N^2 x_1^2 x_2^2$$

Both *samāsa-bhāvanā* and *antara-bhāvanā* are explained by

adding $\pm 2Nx_1x_2y_1y_2$ to both sides

$$Nx_1^2 y_2^2 + Nx_2^2 y_1^2 \pm 2Nx_1x_2y_1y_2 + k_1k_2 = y_1^2 y_2^2 + N^2x_1^2x_2^2 \pm 2Nx_1x_2y_1y_2$$

i.e. $N(x_1y_2 \pm y_1x_2)^2 + k_1k_2 = (y_1y_2 \pm Nx_1x_2)^2$

$$NX^2 + K = Y^2$$

where X (*kaniṣṭha*) $= (x_1y_2 \pm x_2y_1)$

Y (*jyeṣṭha*) $= Nx_1x_2 \pm y_1y_2$

K (*kṣepa*) $= k_1k_2$

Kṛṣṇa further observes (*BP.* p. 134) : एवं खण्डक्षोदेन बहुविधा उपपत्तय: सन्ति । ग्रन्थविस्तरभयान्न लिख्यन्ते । – Thus there are many kinds of *upapattis* for the above[4] (according to the way to substitute for terms of equations). They have not been dealt with, for fear of lengthening the text.

4.3.2. *Upapatti* 2 :

Kaniṣṭha or lesser root $x = (x_1y_2 + y_1x_2)$ by assumption. Squaring this and multiplying by N

$$N(x_1y_2 + y_1x_2)^2 = N(x_1^2y_2^2 + y_1^2x_2^2 + 2x_1y_2x_2y_1)$$

$$= Nx_1^2(Nx_2^2 + k_2) + Ny_1^2\frac{(y_2^2 - k_2)}{N} + 2Nx_1y_2x_2y_1 \quad (\text{since } Nx_2^2 + k_2 = y_2^2)$$

4. *Cf. BG*, ed. Sri Jivanatha Jha, *op.cit.*, p. 162. An alternate method is given by the editor. Taking equations 1 and 2

$$\pm k_1 = y_1^2 - Nx_1^2$$
$$\pm k_2 = y_2^2 - Nx_2^2$$
$$\text{so } k_1k_2 = (y_1^2 - Nx_1^2)(y_2^2 - Nx_2^2)$$
$$k_1k_2 = y_1^2y_2^2 - Nx_1^2y_2^2 - Nx_2^2y_1^2 + N^2x_1^2x_2^2 = y_1^2y_2^2 + N^2x_1^2x_2^2$$
$$\pm 2Nx_1y_1x_2y_2 - (Nx_1^2y_2^2 + Nx_2^2y_1^2 \pm 2Nx_1y_1x_2y_2)$$
$$= (Nx_1x_2 + y_1y_2)^2 - N(x_1y_2 + x_2y_1)^2$$

i.e. $k_1k_2 + N(x_1y_2 + x_2y_1)^2 = (Nx_1x_2 + y_1y_2)^2$

$$NX^2 + K = Y^2$$

i.e. X (*kaniṣṭha*) $= (x_1y_2 + x_2y_1)$

Y (*jyeṣṭha*) $= Nx_1x_2 + y_1y_2$

K (*kṣepa*) $= k_1k_2$

$$= N^2x_1^2x_2^2 + Nx_1^2k_2 + y_1^2y_2^2 - k_2y_1^2 + 2Nx_1x_2y_1y_2$$

$$= N^2x_1^2x_2^2 + Nx_1^2k_2 + y_1^2y_2^2 - k_2(Nx_1^2 + k_1) + 2Nx_1y_1x_2y_2$$

$$= N^2x_1^2x_2^2 + Nx_1^2k_2 + y_1^2y_2^2 - k_2Nx_1^2 - k_1k_2 + 2Nx_1y_1x_2y_2$$

$$= N^2x_1^2x_2^2 + y_1^2y_2^2 + 2Nx_1x_2y_1y_2 - k_1k_2$$

$$= (Nx_1x_2 + y_1y_2)^2 - k_1k_2$$

i.e. $N(x_1y_2 + y_1x_2)^2 + k_1k_2 = (Nx_1x_2 + y_1y_2)^2$

i.e. $NX^2 + K = Y^2$

where, X (new *kaniṣṭha*) $= x_1y_2 + y_1x_2$

Y (new *jyeṣṭha*) $= Nx_1x_2 + y_1y_2$

K (new *kṣepa*) $= k_1k_2$

4.3.3. *Upapatti* 3:

Work on the rational solutions of *varga-prakṛti* was done by Śrīpati also. He derived the rational solution without the aid of the "auxiliary equations". Let m be a rational number optionally chosen,

$$N.1^2 + (m^2 - N) = m^2$$

$$N.1^2 - (N - m^2) = m^2$$

Applying *samāsa-bhāvanā* as explained before, we have for the two equations

prakṛti	*kaniṣṭha*	*jyeṣṭha*	*kṣepa*
N	1	m	$m^2 - N$
	1	m	$-(N - m^2)$

$x = 1 \times m + 1 \times m = 2m$; $y = N \times 1 \times 1 + m \times m = N + m^2$; $k = (m^2 - N)^2$

$$N(2m)^2 + (m^2 - N)^2 = (m^2 + N)^2$$

which is the well-known identity[5] : $(a-b)^2 + 4ab = (a+b)^2$

$$\text{Thus} \qquad N\left(\frac{2m}{m^2 - N}\right)^2 + 1 = \left(\frac{m^2 + N}{m^2 - N}\right)^2$$

$$\text{in other words} \qquad x = \frac{2m}{m^2 - N} \qquad y = \frac{m^2 + N}{m^2 - N}$$

Note : This rational solution of the *varga-prakṛti* derived by Śrīpati was rediscovered in Europe by Brouncker[6] in 1657.

4.3.4. *Upapatti* 4 :

[7]In the equation $Nx^2 + 1 = y^2$, assume $x = 2x_1$ since $(a-b)^2 + 4ab = (a+b)^2$, we have $4x_1^2 N + (N - x_1^2)^2 = (N + x_1^2)^2$

i.e. $$N(2x_1)^2 + (N - x_1^2)^2 = (N + x_1^2)^2$$

i.e. $$N\left(\frac{2x_1}{N - x_1^2}\right)^2 + 1 = \left(\frac{N + x_1^2}{N - x_1^2}\right)^2$$ (Assuming $N \neq x_1^2$ since N is not a square number)

So we have for any x, *kaniṣṭha* $= \dfrac{2x_1}{N - x_1^2}$ *jyeṣṭha* $= \dfrac{(N + x_1^2)}{(N - x_1^2)}$ which are rational solutions of $Nx^2 + 1 = y^2$.

Thus there are several proofs for the *bhāvanā* method. Kṛṣṇa corroborates Bhāskara's statement (*BG*.v.83): ततो ज्येष्टमिहानन्त्यं भावनातस्तथेष्टत: ।। – that by the principle of *samāsa-bhāvanā* or *antara-bhāvanā,* and according to the optional number, an infinite number of solutions can be obtained (*BP.* p. 136) : इह कनिष्ठज्येष्टयोर्भावनावशात्तथेष्टवशादानन्त्यम् अस्ति ।

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5. *BP.* p. 136 : चतुर्गुणस्य घातस्य युतिवर्गस्य चान्तरं राश्यन्तर कृतेस्तुल्यमिति राश्यन्तरवर्गेण युतश्चतुर्गुणितो घाते युतिवर्गो भवति ।
6. Datta and Singh, *op.cit.*, Vol. II, p. 155.
7. Different *upapattis* for the above equation $Nx^2 + 1 = y^2$ given by modern scholars are given in the Appendix II.

Remarks : Datta and Singh observe : "Modern historians are incorrect in stating that Fermat (1657) was the first to assert that the equation $Nx^2 + 1 = y^2$, where N is a non-square integer, has an unlimited number of solutions in integers. The existence of an infinite number of integral solutions was clearly mentioned by Hindu algebraists long before Fermat"[8]

Brahmagupta's *bhāvanā* was rediscovered by the great Swiss mathematician Euler (1707 – 1783) in 1764 CE.[9] Euler highlighted the result in his writings as '*theorema eximium*' (a theorem of capital importance), '*theorema elegantissum*' (most elegant theorem) etc.

Bhāskara was evidently aware of Brahmagupta's *bhāvanā*. He used the same word *bhāvanā,* but *vajrābyāsa* instead of *vajravadha* used by Brahmagupta.

4.4. Comparison between European and Indian methods :

Two examples[10] are given to facilitate comparison of the European method[11] with the Indian method.

<u>Example 1 :</u> Find a particular solution of the equation $21x^2 + 1 = y^2$.

European Method :

Here $N = 21$, and the infinite continued fraction expansion of $\sqrt{21}$ is

$\sqrt{21} = \left\{4,\overline{1,1,2,1,1,8}\right\} = \left\{a_1, \overline{a_2\ a_3\ a_4\ a_5\ a_6, 2a_1}\right\}$ (a_1, a_2 ... are the partial quotients of the continued fraction expansion and c_1, c_2 ... are the convergents), and the overhead bar indicates infinite recurrence.

8. Datta and Singh, *op.cit.,* Vol. II, p. 150.

9. *ibid.*, p. 148.

10. Adapted from C.D.Olds, *Continued Fractions*, California, 1961, p. 117.

11. See Appendix III, for details of the procedure of getting the periodic continued fraction for a surd $\sqrt{N}$ and its relationship to the equation $Nx^2 + 1 = y^2$.

Here the penultimate convergent in the recurrence is the n^{th} convergent, for $n = 6$, which is even. By calculation, convergent $c_6 = \frac{55}{12}$ so that $x_1 = 12$; $y_1 = 55$ is one solution. To verify this :

$$21\ (12^2) + 1 = 55^2$$

$x = 12,\ y = 55$ is a particular solution of $21x^2 + 1 = y^2$.

Indian Method :

$$21x^2 + 1 = y^2$$

In this equation *prakṛti* is 21, *kaniṣṭha* is x, *jyeṣṭha* is y and *kṣepa* is 1.

By observation, the auxiliary equation is

$$21(1)^2 + 4 = (5)^2 \text{ where } x = 1 \quad y = 5 \quad k = 4$$

Using the *sūtra* (*BG.* v. 82) : इष्टवर्गहृत: क्षेप: क्षेप: स्यादिष्टभाजिते ।

$$21\left(\frac{1}{2}\right)^2 + \frac{4}{4} = \left(\frac{5}{2}\right)^2$$

i.e.

$$21\left(\frac{1}{2}\right)^2 + 1 = \left(\frac{5}{2}\right)^2$$

We have the following columns for *bhāvanā* process to get integral solutions :

prakṛti	*kaniṣṭha*	*jyeṣṭha*	*kṣepa*
21	$\frac{1}{2}$	$\frac{5}{2}$	1
	$\frac{1}{2}$	$\frac{5}{2}$	1

So, *kaniṣṭha* $x_1 = \frac{1}{2} \times \frac{5}{2} + \frac{1}{2} \times \frac{5}{2} = \frac{5}{2}$

jyeṣṭha $y_2 = 21 \times \frac{1}{2} \times \frac{1}{2} + \frac{5}{2} \times \frac{5}{2} = \frac{23}{2}$

Still the roots are not integral. So *bhāvanā* has to be repeated.

$$\begin{array}{cccc} 21 & \frac{5}{2} & \frac{23}{2} & 1 \\ & \frac{1}{2} & \frac{5}{2} & 1 \end{array}$$

So, *kaniṣṭha* $x_2 = \frac{5}{2} \times \frac{5}{2} + \frac{23}{2} \times \frac{1}{2} = 12$

jyeṣṭha $y_2 = 21 \times \frac{5}{2} \times \frac{1}{2} + \frac{23}{2} \times \frac{5}{2} = 55$

kṣepa $= 1 \times 1 = 1$

Thus simple arithmetic operations have yielded the integral solution.

Example 2 : Find a particular solution of the equation $29x^2 + 1 = y^2$

The European Method :

The expansion of $\sqrt{29}$ as continued fraction is

$$\sqrt{29} = \left\{5, \overline{2,1,1,2,10}\right\} = \left\{a_1, \overline{a_2, a_3, a_4, a_5, 2a_1}\right\}$$

Here the penultimate convergent corresponds to the rank $n = 5$ which is an odd number. The first five convergents are respectively $\frac{5}{1}, \frac{11}{2}, \frac{16}{3}, \frac{27}{5}, \frac{70}{13}$

Though one may expect $x_1 = 13$, $y_1 = 70$ to be solutions, $29(13)^2 + 1 \neq 70^2$. Hence the convergents corresponding to the next period are to be found out. These are :

$$\frac{727}{135}, \frac{1524}{283}, \frac{2251}{418}, \frac{3775}{701}, \frac{9801}{1820}\ ; \ c_{10} = \frac{9801}{1820}$$

If we take $x_2 = 1820$, $y_2 = 9801$ then $29(1820)^2 + 1 = (9801)^2$. Therefore $x = 1820$, $y = 9801$ is a particular solution of the equation $29x^2 + 1 = y^2$.

Indian Method :

The equation given is $29x^2 + 1 = y^2$. Here *prakṛti* is 29, *kaniṣṭha* is x and *jyeṣṭha* is y. On observation, the auxiliary equation can be written as

$29(1)^2 - 4 = 5^2$. Where $x = 1 \quad y = 5 \quad k = -4$. Dividing throughout by 4

$$29\left(\frac{1}{2}\right)^2 - \frac{4}{4} = \left(\frac{5}{2}\right)^2$$

i.e. $$29\left(\frac{1}{2}\right)^2 - 1 = \left(\frac{5}{2}\right)^2$$

The columns for *bhāvanā* are

$$\begin{array}{cccc} 29 & \frac{1}{2} & \frac{5}{2} & -1 \\ & \frac{1}{2} & \frac{5}{2} & -1 \end{array}$$

These yield

$$\textit{kaniṣṭha } x_1 = \frac{1}{2} \times \frac{5}{2} + \frac{1}{2} \times \frac{5}{2} = \frac{5}{2}$$

$$\textit{jyeṣṭha } y_1 = 29 \times \frac{1}{2} \times \frac{1}{2} + \frac{5}{2} \times \frac{5}{2} = \frac{27}{2}$$

$$\textit{kṣepa } k_1 = -1 \times -1 = +1$$

Bhāvanā is done again to get integral roots.

$$\begin{array}{cccc} 29 & \frac{1}{2} & \frac{5}{2} & -1 \\ & \frac{5}{2} & \frac{27}{2} & +1 \end{array}$$

We get now

$$\textit{kaniṣṭha } x_2 = \frac{1}{2} \times \frac{27}{2} + \frac{5}{2} \times \frac{5}{2} = 13$$

$$\textit{jyeṣṭha } y_2 = 29 \times \frac{1}{2} \times \frac{5}{2} + \frac{5}{2} \times \frac{27}{2} = 70$$

$$\textit{kṣepa } k_2 = -1 \times +1 = -1$$

So $x_2 = 13$, $y_2 = 70$ are the roots of the equation $29(13)^2 - 1 = 70^2$.

To get the roots of the required equation, *bhāvanā* is used again

29	13	70	−1
	13	70	−1

$$\therefore \quad \textit{kaniṣṭha}\ x_3 = 13 \times 70 + 13 \times 70 = 1820$$

$$\textit{jyeṣṭha}\ y_3 = 29 \times 13^2 + 70^2 = 9801$$

Note : Repetition of simple arithmetic operations, has yielded the solutions from the known solutions of another equation.

Brahmagupta's partial solution, apart from being a remarkable landmark by itself, was also a significant step towards the celebrated *cakravāla* algorithm. The *cakravāla* is a perfect method, free from trial and error unlike in the European method where we have to take a suitable convergent at the end of the period for obtaining, for any N all positive integral solutions of $Nx^2 + 1 = y^2$; this is discussed in the following sections.

4.5. *Cakravāla*:

The *cakravāla* method, its evolution and other forms of *varga-prakṛti* are discussed here.

4.5.1. Evolution :

The equation $Nx^2 + 1 = y^2$ (erroneously called Pell's equation) has a long and interesting history. The earliest civilizations of the world, the Greek and the Indian were fascinated by the problem. While the Greeks never gave more than particular solutions for cases like $2x^2 \pm 1 = y^2$, the Indians were the first "to realise the true inwardness of the problem and to give a general solution based on the principle of composition of quadratic forms"[12].

12. A.A.K. Ayyangar, "New Light on Bhaskara's Chakravala or Cyclic Method of solving Indeterminate Equations or the Second Degree in two Variables" *Journal of Indian Mathematical Society 18* (1920-30), p. 225.

According to K.S. Shukla, the earliest author who refers to *cakravāla* is Udayadivākara (11th Cent. A.D.) who quotes Jayadeva's verses on *cakravāla* in his text *Sundarī.* But there is no information regarding the exact date or other results of Jayadeva.[13] It is surprising that Jayadeva is not mentioned by Bhāskara ; infact, he does not seem to have been referred to by most of the mathematicians. However, from Bhāskara's own words *cakravālamidam jaguḥ* we may deduce that it was known prior to him.

At the end of his *BG,* Bhāskara also makes a general acknowledgement of Brahmagupta, Śrīdhara and Padmanābha (*BG.* p. 219) : ब्रह्माह्वयश्रीधरपद्मनाभबीजानि यस्मादतिविस्तृतानि । Since Brahmagupta does not mention *cakravāla*, there is a strong possibility that at least, Padmanābha or Śrīdhara, wrote on *cakravāla.* Unfortunately their works are lost to us; some of the contents of Śrīdhara's work are known only through Bhāskara's own reference to them (*BG.* pp. 60-1).

Remarkable success was achieved by Bhāskara when he developed a simple method to derive the auxiliary equation. This equation would have the required *kṣepas* ± 1, ± 2 or ± 4 simultaneously with two integral solutions from any auxiliary equation empirically formed with any simple value of the *kṣepa.* This method is famous *cakravāla* method or cyclic method, so named for its iterative character.

4.5.2. Definition of *cakravāla* :

The term *cakravāla* means 'circle'[14]. Sūryadāsa rightly observes that, because the same set of operations proceed as in a circle, being

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13. K.S. Shukla in his article, 'Ācārya Jayadeva, the Mathematician', *Gaṇita* 5, No.1 (June 1954), p. 19. fn. 3.

14. Datta and Singh, *op.cit.*, Vol. II, p. 162, fn.1.

applied again and again in a continuous round, the method is called *cakravāla.*[15]

4.5.3. Bhāskara's *cakravāla* :

Bhāskara's *cakravāla* is based on the following Lemma :

Lemma : If $Na^2 + k = b^2$ is an auxiliary equation where a, b, k are integers, k being positive or negative then,

$$N\left(\frac{am+b}{k}\right)^2 + \frac{m^2 - N}{k} = \left(\frac{bm + Na}{k}\right)^2$$

where m is any arbitrary whole number.

Proof : The rationale behind this is simply the following : Consider the equations

$Na^2 + k = b^2$ and

$N(1)^2 + m^2 - N = m^2$ (Śrīpati's auxilary equation **4.3.3.**, *Upapatti* 3).

Then using Brahmagupta's *bhāvanā,*

prakṛti	*kaniṣṭha*	*jyeṣṭha*	*kṣepa*
N	a	b	k
	1	m	$m^2 - N$

$$kaniṣṭha = a \times m + b \times 1 = am + b$$

$$jyeṣṭha = N \times a \times 1 + b \times m = Na + mb$$

$$kṣepa = k\,(m^2 - N)$$

Thus $N(am + b)^2 + k(m^2 - N) = (Na + mb)^2$

dividing by k^2

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15. Sūryadāsa's *Sūryaprakaśa*, British Library, San. I.O. 1533a, fol. 25 : ह्रस्वज्येष्ठमूलाभ्यां कुट्टकः कृतः कुट्टकाद्गुणलब्धिभ्यां च पुनः कुट्टक इत्येतस्य विधेर्मण्डलाकृतिकत्वेन एवमिदं चक्रवालमिति । ह्रस्वज्येष्ठमूलाभ्यां कुट्टकः कृतः कुट्टकाद्गुणलब्धिभ्यां च पुनः कुट्टक इत्येतस्य विधेर्मण्डलाकृतिकत्वेन आद्यगणकैश्चक्रवालमिति कल्पिता । चक्रवालं तु मण्डलमिति अमरोक्तेः ।

Also *Cf.* Jivanatha Jha in his edition of *BG.* p. 175 : चक्रो रथाङ्गं तदिव वलते परिवर्तते – इति चक्रवालः । अर्थादत्र कुट्टकवर्गप्रकृत्योश्चक्रवद्भ्रमणं कुट्टकाद् वर्गप्रकृतेः ततः कुट्टकस्य चावसरप्रसंगदर्शनात् ।

$$N\left(\frac{am+b}{k}\right)^2 + \frac{m^2-N}{k} = \left(\frac{Na+mb}{k}\right)^2 \qquad ①$$

m can be so chosen such that $\frac{am+b}{k}$ is an integer since its value can be determined by means of *kuṭṭaka*, *viz.*, by solving the equation $ax+b=ky$ in integers and taking the solution for x as m. Obviously there can be infinite number of values for m. But Bhāskara says m should be so chosen as to make $|m^2-N|$ minimum. If $\frac{m^2-N}{k}$ is equal to ± 1, ± 2, ± 4, then Brahmagupta's *bhāvanā* can be applied immediately. If $\frac{m^2-N}{k}$ is not one of the above values then *kuṭṭaka* is performed again and again till $\frac{m^2-N}{k} = \pm 1, \pm 2, \pm 4$ is reached. Bhāskara is aware that this process will end after a finite number of steps.

Let $\frac{am+b}{k} = a_1$, $\frac{m^2-N}{k} = k_1$, $\frac{bm+Na}{k} = b_1$. Then equation ① becoms $Na_1^2 + k_1 = b_1^2$; from this a new equation $Na_2^2 + k_2 = b_2^2$ can be obtained proceeding in the same manner.

4.5.3.1. Steps involved :

There are bascially four steps involved in the *cakravāla* method, for the equation be $Na^2 + k = b^2$.

Step 1 : (*BP.* p. 139) : क्रमेण ह्रस्वज्येष्ठक्षेपान् भाज्यक्षेपभाजकान् कृत्वा । Form the *kuṭṭaka* equation with a, k, b. so that a is the dividend, b is the *kṣepa* or additive and k is the divisor.

Step 2 : (*BP.* p. 139) : गुणस्य वर्गे प्रकृतितश्च्युते प्रकृत्योने वा शेषमल्पं स्यात् । यस्य गुणस्य वर्गेण प्रकृत्या सहान्तरं कृतं तस्य गुणस्य या लब्धिः तत्कनिष्ठं पदं स्यात् । – From the solutions m, a_1 of this *kuṭṭaka*, m is so chosen as to make $|m^2-N|$ minimal. The quotient a_1 corresponding to that value of m will be new *kaniṣṭha* root.

Step 3 : (*BP.* p. 139) : तत्र शेषं पूर्वक्षेपहृतं सत् क्षेप: स्यात् । गुणवर्गे प्रकृतितश्च्युते सति अयं क्षेपो व्यस्त: स्यात् । धनं चेत् ऋणं, चेद्धनं भवेदित्यर्थ: । – The difference $(m^2 - N)$ divided by k gives the new *kṣepa* k_1, k_1 being positive or negative according as the difference $(m^2 - N)$ is positive or negative.

Step 4 : Finally the new greater root or *jyeṣṭha* root has to be found. (*BP.* p. 139) : तत: कनिष्ठाज्जेष्ठं पूर्ववत् स्यात् – Once *kaniṣṭha* is available, *jyeṣṭha* is the square root of "square of *kaniṣṭha* multiplied by *prakṛti* " and the new *kṣepa* added to it. Therefore $Na_1^2 + k_1$ should give b_1^2 and thence b_1.

Step 4a : Kṛṣṇa's additional rule is (*BP.* p. 142) : अन्यथापि ज्येष्ठापेक्षा चेत्तदा गुणकगुणितं ज्येष्ठं प्रकृतिगुणेन कनिष्ठेन युतं क्षेपभक्तं ज्येष्ठं भवतीत्यस्मदुक्तमार्गेण ज्येष्ठं कुर्यात् । – The original greater root multiplied by the multiplier is added to the lesser root multiplied by *prakṛti* and the sum divided by the *kṣepa* gives the greater root. In other words $b_1 = \frac{bm + Na}{k}$ as evident from above.

Remarks : While Bhāskara says : तत: कनिष्ठाज्जेष्ठं पूर्ववत् स्यात् – Kṛṣṇa has contributed the above new rule to get *jyeṣṭha* thereby avoiding finding square root of large numbers[16].

4.5.4. Bhāskara's remarks on *alpam śeṣakam yathā* :

Bhāskara stipulates that m should be so chosen as to make $\left| m^2 - N \right|$ the least. Kṛṣṇa explains (*BP.* p. 142) : गुणवर्गे प्रकृत्या ऊने अथवा अल्पं शेषकं यथा तत्तु क्षेपहृतं क्षेप इति । तत्र प्रकृतितश्चेद् गुणवर्गो अधिको भवति तदैव क्षेपभक्तं गुणवर्गप्रकृत्यन्तरयोर्ज्यं

16. Datta and Singh (*op.cit*, Vol.II, p.165), refer to Nārāyaṇa giving a different rule for the same. The *kaniṣṭha* multiplied by the multiplier and diminished by the product of the previous *kaniṣṭha* and new *kṣepa* will be the new *jyeṣṭha*. That is $b_1 = a_1 m - k_1 a$.

Cf. Selenius, "Rationale of the Chakravala process of Jayadeva and Bhaskara II" *op.cit.*, p. 174.

क्षिप्तस्य न्यूनत्वात् । यदा तु गुणवर्गो न्यूनः तदा क्षेपभक्तं गुणवर्गप्रकृत्यन्तरशोध्यं क्षिप्तस्य अधिकत्वात् । अत उक्तं व्यस्तः प्रकृतितः च्युतः इति । यत्तु गुणवर्गप्रकृत्योरन्तरमल्पं यथा स्यात्तथा गुणः कल्प्य इत्युक्तं तत्क्षेपस्य लघुत्वार्थम् ॥ – Let the solutions of the *kuṭṭaka* be m, a_1. m should be chosen so that $m^2 - N$ is the least. $m^2 - N$ divided by k_1 the first *kṣepa* gives k_2 the second *kṣepa*. Now $m^2 - N$ is positive if $m^2 > N$. If $N > m^2$, then $m^2 - N$ would be negative. Then, the sign of the second *kṣepa* should be reversed. The new *kaniṣṭha* will be a_1 corrosponding to this m.

In the infinite system of values, there should be a set of two integers, one less then $\sqrt{N}$, the other greater than $\sqrt{N}$ in the immediate neighbourhood of $+\sqrt{N}$ and similarly two in the immediate neighbourhood of $-\sqrt{N}$. The squares of these four integers are evidently nearer to N than the squares of any other value of m in the equation $am + b = ka_1$. According to Bhāskara, we have to choose that m whose square is closest to N.

In this connection, A.A.K. Ayyangar makes a note that this rule has exceptions and records : "An exceptional case may occur when the root corresponding to the nearest square leads back to the previous step in the process of reduction. In this case, the root corresponding to the nearest of the remaining squares should be chosen"[17]. This exceptional case has not been explicitly noted by Bhāskara. However, A.A.K. Ayyangar feels that Bhāskara has provided the example $61x^2 + 1 = y^2$, only to explain the above exceptional case.[18]

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17. A.A.K. Ayyangar, "New Light on Bhaskara's Chakravala or Cyclic Method of solving Indeterminate Equations of the Second Degree in two Variables", *op.cit.*, p.235
18. *ibid.*, p.235 fn.

4.6. The two theorems deduced from Bhāskara's cyclic method :

Modern scholars have deduced two theorems from the cyclic method of Bhāskara[19] :

1) when a_1 is an integer, k_1 and b_1 are each a whole number (proof given by Hankel) ;

2) his cyclic method will in every case lead to the desired result (proof given by A.A.K. Ayyangar).

4.6.1. Theorem 1 :

The first theorem is not new. Kṛṣṇa has stressed in more than one place that for the sake of integral solutions, the method of *kuṭṭaka* is used. Kṛṣṇa explains why *kuṭṭaka* is resorted to (*BP.* p. 142) : अत्र यद्यपि इष्टवशादेव पदसिद्धिरस्तीति कुट्टकस्य नापेक्षा तथापि अभिन्नत्वार्थं कुट्टकः कृतः । – *Kuṭṭaka* is used only to arrive at integral solutions. While introducing the *cakravāla* method Kṛṣṇa has used the phrase (*BP.* p. 139) : कनिष्ठज्येष्ठयोरभिन्नतार्थं । This is only emphasizing Bhāskara's *sūtra* where he says (*BG.* v. 88) चतुर्द्व्येकयुतावेवमभिन्ने भवतः पदे ।। चतुर्द्विक्षेपमूलाभ्यां रूपक्षेपार्थभावना । – In order to derive integral roots corresponding to an equation with unity as *kṣepa* from those of the equation with the *kṣepa* $-1, \pm 2, \pm 4$, the principle of *bhāvanā* is applied.

4.6.1.1. Hankel's Proof[20] for Theorem 1 :

According to Bhāskara's *cakravāla*, if $Na^2 + k = b^2$, then

$$N\left(\frac{am+b}{k}\right)^2 + \frac{m^2-N}{k} = \left(\frac{bm+Na}{k}\right)^2$$ where m is any chosen number.

19. Datta and Singh, *op.cit.*, Vol. II, p.171

20. See, Datta and Singh, *op.cit.*, Vol. II, pp. 171-72

Thus the second set of roots are :

$$a_1 = \frac{am+b}{k} \qquad b_1 = \frac{bm+Na}{k} \qquad \text{and} \qquad k_1 = \frac{m^2-N}{k}$$

To prove that if a_1 is an integer, then b_1 and k_1 are also integers. Since

$a_1 = \dfrac{am+b}{k}$, we have $a_1 k = am + b$. Also $k = b^2 - Na^2$.

So, $a_1(b^2-Na^2) = am + b$

i.e. $a_1 b^2 - b = Na_1 a^2 + am$

i.e. $b\,(a_1 b - 1) = a\,(Naa_1 + m)$

i.e. $\dfrac{b}{a}\,(a_1 b - 1) = Naa_1 + m$

a, b have no common factor. This shows that 'a' must divide $(a_1 b - 1)$

or $\dfrac{(a_1 b - 1)}{a}$ should be an integer. ①

Also we have

$$a_1 = \frac{am+b}{k} \quad\Rightarrow\quad a_1 k = am + b$$

$$b_1 = \frac{Na+mb}{k} \quad\Rightarrow\quad b_1 k = Na + mb.$$

$$a_1 k = am + b \quad\Rightarrow\quad am = a_1 k - b$$

$$m = \frac{a_1 k - b}{a} \qquad ②$$

$$b_1 k = Na + mb \quad\Rightarrow\quad m = \frac{b_1 k - Na}{b} \qquad ③$$

Equating ② and ③ we have

$$\frac{a_1 k - b}{a} = \frac{b_1 k - Na}{b}$$

$$\text{i.e. } b(a_1 k - b) = a(b_1 k - Na)$$

$$kba_1 - b^2 = kab_1 - Na^2$$

$$kba_1 - kab_1 = b^2 - Na^2 = k$$

Dividing by k throughout $ba_1 - ab_1 = 1$

i.e. $ab_1 = ba_1 - 1 \Rightarrow b_1 = \dfrac{ba_1 - 1}{a}$

So, b_1 is an integer from ①. Now,

$$m^2 - N = \left(\frac{a_1 k - b}{a}\right)^2 - N$$

$$= \frac{a_1^2 k^2 + b^2 - 2a_1 kb}{a^2} - N$$

$$= \frac{a_1^2 k^2 - 2a_1 kb - Na^2 + b^2}{a^2}$$

$$= \frac{a_1^2 k^2 - 2a_1 kb + k}{a^2} \qquad (\text{since } b^2 - Na^2 = k)$$

$$= \frac{k}{a^2}\,(a_1^2 k - 2a_1 b + 1)$$

Therefore $\dfrac{k}{a^2}\,(a_1^2 k - 2a_1 b + 1)$ is a whole number. Since a^2, k have no common factor, a^2 divides $(a_1^2 k - 2a_1 b + 1)$. So

$$\frac{k}{a^2}\,\frac{(a_1^2 k - 2a_1 b + 1)}{k} = \frac{m^2 - N}{k} = k_1 = \text{an integer.}$$

Or else

$$k_1 = \frac{m^2 - N}{k} = \frac{a_1^2 k - 2a_1 b + 1}{a^2}$$

$$= \frac{a_1^2 (b^2 - Na^2) - 2a_1 b + 1}{a^2} = \frac{a_1^2 b^2 - 2a_1 b + 1}{a^2} - \frac{Na^2 a_1^2}{a^2}$$

$$= \frac{(a_1 b - 1)^2}{2} - Na_1^2$$

$$= b_1^2 - Na_1^2 .$$

So that k_1 is an integer.

4.6.2. Theorem 2 :

Theorem 2 is proved by A.A.K. Ayyangar,[21] establishing the fact that the *cakravāla* must end after a finite number of steps.

Remarks : A.A.K. Ayyangar (p. 231) clearly points out that : "as T.L. Heath says, nothing is wanting to the cyclic method except the proof that it will in every case lead to the desired result, whenever N is a non-square number ; but he is wrong in supposing that Lagrange was the first to supply the proof for Bhāskara's cyclic method".

According to Andre Weil, "to have developed the *cakravāla* and to have applied it successfully to such difficult numerical cases such as $N = 61$ or $N = 67$ had been no mean achievement".[22]

4.6.3. The equation $Nx^2 + 1 = y^2$:

There is an interesting story[23] behind the equation of the type $Nx^2 + 1 = y^2$. In 1657 the famous French mathematician Fermat sent a

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21. Refer A.A.K. Ayyangar's article, "New Light on Bhaskara's Chakravala or Cyclic Method of solving Indeterminate Equations of the Second Degree in two Variables", *op.cit.* pp. 231-33, for the proof not reproduced here.
22. Andre Weil, *Number Theory: An approach through History from Hammurapi to Legendre*, Birkhauser, 1984, p. 21.
23. Adapted from Article "How to solve Pell's equation" by J.J.O'Connor and E.F. Robertson, 2002, www.history.mcs.st-andrews.ac.uk/HistTopics/Pell.html

public challenge to his friend, Bernard Frenicle de Bessy and then on to Brouncker and Wallis in England to solve the equation $61x^2 + 1 = y^2$ in integers. "We await", he challenged, "the solutions which, if England or Belgian and Celtic Gauls cannot give them, Narbonian Gaul will . . . "[24]. Narbonian Gaul was of course the area where Fermat lived! None of these succeeded in solving the equation. It was only in 1732 that the renowned mathematician Euler gave a complete solution. But remarkably, the very same equation had been dealt with and solved in a few steps by Bhāskara II by the famous *cakravāla* method more than five centuries earlier. Bhāskara gave the least solution as $x = 226153980$ and $y = 1766319049$. No wonder Andre Weil exclaims "what would have been Fermat's astonishment if some missionary, just back from India had told him that his problem had been successfully tackled there by native mathematicians almost six centuries earlier".[25]

Several mathematicians participated in Fermat's challenge. Brouncker came out with a method of solution which was essentially the same as the method of continued fractions. When challenged by Frenicle de Bessy to solve $313x^2 + 1 = y^2$. Brouncker gave as his smallest solution $x_1 = 1819380158564160$ $y_1 = 32188120829134849$ adding that it had taken him "an hour or two" to find. Unfortunately Brouncker did not do any further work on this subject. Still it "was he who first found a general and algebraic procedure for solving $Nx^2 + 1 = y^2$ and laid a foundation on which future number theorists were to build (their work)."[26]

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24. Jacqueline A. Stedall, "Catching Proteus: The Collaborations of Wallis and Brouncker : II. Number Problems", *Notes and Records of the Royal Society of London*, Vol. 54(3), (2000), p. 318.
25. Andre Weil, *op.cit.*, p.81.
26. Jacqueline A. Stedall, *op.cit.*, p. 327

Later in the 18th Century A.D., Euler gave Brahmagupta's Lemma and its proof which is similar to what has been given earlier. He was aware of Brouncker's work on Pell's equation as presented by Wallis, but he was totally unaware of the contributions of the Indian mathematicians. He gave the basis for the continued fraction approach to solving Pell's equation which was put into a polished form by Lagrange in 1766. The other major contribution of Euler was in wrongly naming the equation as "Pell's equation" thinking that the major contributions which Wallis had reported on, as due to Brouncker, were infact the work of Pell.

Lagrange published his *Additions to Euler's Elements of Algebra* in 1771, and this contains his rigorous version of Euler's continued fraction approach to Pell's equation.

4.7. Comparison between *Cakravāla* and Lagrange's Method :

Lagrange's "Additions" established for the first time that for every N, Pell's equation had infinitely many solutions. This is evident in Bhāskara's *cakravāla* because whenever *kṣepa* is ± 1, ± 2, ± 4, he resorts to Brahmagupta's *bhāvanā* to get the desired result and by the *sūtra* (*BG.* v. 83ef) : ततो ज्येष्ठमिहानन्त्यं भावनातस्तथेष्टत: ।। it is clear that the solutions are infinite.

The solution by Lagrange's method depends on the continued fraction expansion of $\sqrt{N}$. In the expansion of the continued fraction of the square root of an integer, the same denominators recur periodically,

$$\sqrt{N} = \left\{a_1 \overline{a_2\, a_3 \ldots a_k}\right\} = a_1 + \frac{1}{a_2 +}\ \frac{1}{a_3 +}\ \ldots$$

where $a_1 a_2 \ldots a_k$ is the first period ($a_k = 2a_1$). The first (least) trivial soluion to the equation $Nx^2 + 1 = y^2$ is given by the penultimate

convergent p_{k-1} / q_{k-1} when k is even or p_{2k-1} / q_{2k-1} when k is odd, and $x = q_i$, $y = p_i$, $i = k-1$ or $2k-1$ as the case may be.

For example the 'regular'[27] expansion of $\sqrt{58} = \{7, \overline{1, 1, 1, 1, 1, 1, 14}\}$ $p_1 / q_1 = 7 / 1$, $p_2 / q_2 = 8 / 1 \ldots p_{13} / q_{13} = 19603/2574$ i.e. $x = 2574$, $y = 19603$. in $58x^2 + 1 = y^2$.

$\sqrt{58}$ can also be written as a 'half regular' expansion (denominators with negative numerators) $\{8, \overline{2, 1, 1, 1, 1, 15}\}$ $p_1 / q_1 = 8 / 1 \ldots$ $p_{11} / q_{11} = 19603 / 2574$. $\sqrt{58}$ can also be written as $\{8, \overline{3, 2, 2, 16}\}$ $p_1 / q_1 = 8 / 1 \ldots p_8 / q_8 = 19603 / 2574$. So the eighth convergent produces the required result.

Using *cakravāla*, the method is shorter. Since p_4 / q_4 gives $99 / 13$ which makes $58(13)^2 - 1 = 99^2$, *bhāvanā* can be immediately used to get the desired result.

4.8. *Cakravāla* method :

Let the equation to be solved be $58x^2 + 1 = y^2$

Take equation $58(x)^2 + k = y^2$

Choose x, k such that left hand side (l.h.s.) is a square. The auxiliary equation is

$$58(1)^2 + 6 = 8^2$$

The corresponding *kuṭṭaka* $x + 8 = 6y$

The solution for the above *kuṭṭaka* are (4, 2) (10, 3) (16, 4) ...

27. The expressions 'regular' and 'half regular' expansion are used by C.O. Selenius in his article "Rationale of the Chakravala process of Jayadeva and Bhaskara II," *op.cit.* pp. 167 - 184. See Appendix III for expansion of $\sqrt{N}$.

Choose (10, 3) which gives the least value for $\frac{m^2 - N}{k}$ *viz.*, $\frac{10^2 - 58}{6} = 7$

(10,3) is chosen because by choosing (4,2) we go back to *kuṭṭaka*[28] and eventually arrive at 'm' = 10. (see **4.5.3.** for notation).

Now take $x = 3$, and the equation is $58(3)^2 + 7 = 23^2$

The corresponding *kuṭṭaka* is $3x + 23 = 7y$ following the same procedure as in the first step. Solution sets for this *kuṭṭaka* are (4, 5) (11, 8) . . .

Choosing (4, 5), $\frac{58 - 4^2}{7} = 6$ with negative sign, since if *kṣepa* 7 has (+) sign, *kṣepa* 6 should have (–) sign. The next choice of x is $x = 5$ and the corresponding relation is $58\,(5)^2 - 6 = 38^2$

From *kuṭṭaka* $5x + 38 = 6y$, we get the solution sets (2, 8) (8, 13) (14, 18) . . . Choosing (8, 13), since $\frac{58 - 2^2}{6} > \frac{8^2 - 58}{6}$, $\frac{14^2 - 58}{6} > \frac{8^2 - 58}{6}$

$$\frac{8^2 - 58}{6} = -1$$

$$58\,(13)^2 - 1 = 99^2$$

Using *bhāvanā* with the columns

prakṛti	*kaniṣṭha*	*jyeṣṭha*	*kṣepa*
58	13	99	–1
	13	99	–1

we have

28. See **4.5.4** for A.A.K. Ayyangar's comments on the same.

$$x = 2 \times 13 \times 99 = 2574$$

$$y = 58 \times 13 \times 13 + 99 \times 99 = 19603$$

$$k = -1 \times -1 = +1$$

i.e. $58\,(2574)^2 + 1 = (19603)^2$.

Thus the solution is arrived in a few steps in the Indian method whereas the European method may involve more than the first period of continued fraction.

4.8.1. Superiority of the *cakravāla* method :

The charts given below illustrate the superiority of the *cakravāla* method. The first chart gives the number of steps required to solve the *varga-prakṛti*, using *bhāvanā* ; the second without *bhāvanā* and the third, the European method.

The equation taken is $58x^2 + 1 = y^2$. Let x_1, y_1 denote values of x and y satisfying the equation $58x^2 + k_1 = y^2$, for some $k = k_1$.

1. *Cakravāla* with *bhāvanā* :

	S.No.	x_1	k_1	y_1
5 steps	1	1	+6	8
	2	3	+7	23
	3	5	−6	38
	4	13	−1	99
	5	2574	+1	19603

The fifth step yields the desired solution $58(2574)^2 + 1 = (19603)^2$

2. *Cakravāla* without *bhāvanā* :

	S.No.	x_1	k_1	y_1
	1	1	+6	8
	2	3	+7	23
	3	5	−6	38
	4	13	−1	99
8 steps	5	203	−6	1546
	6	596	−7	4539
	7	1585	−9	12071
	8	2574	+1	19603

Though the *kṣepa* is −1, in the 4th step, *bhāvanā* has not been used and so, the solution is arrived at in the 8th step where $k_1 = +1$ to have

$$58(2574)^2 + 1 = (19603)^2$$

3. European Method : For convenience we construct a table to indicate that $58p_i^2 + k_i = q_i^2$, $i = 1, 2, 3 \ldots 13$. For the calculation of p_i and q_i , $i = 1, 2, 3 \ldots 13$ see Appendix V.

	S.No.	x_1	k_1	y_1
	1	1	6	8
	2	2	−7	15
	3	3	+7	23
	4	5	−6	38
	5	8	+9	61
	6	13	−1	99
13 steps	7	190	+9	1447
	8	203	−6	1546

9	393	+7	2993
10	596	−7	4539
11	989	+6	7352
12	1585	−9	12071
13	2574	+1	19603

The 13th step yields $58(2574)^2 + 1 = (19603)^2$

Thus it transpires that the *cakravāla* method has a fewer number of steps than the European method, especially when used along with *bhāvanā* .

We point out the following observations made by C.O. Selenius in his special study on *cakravāla.* They are as follows :

1. The pattern in the recurring sequences both in the *cakravāla* and the quotients in the Europeon method is palindromic, i.e., the sequence is of two parts where the second half of the periodic sequence is the first half in reverse. The last number in the repeating sequence is double the integer part of the square root. In the example shown above, for instance, $58x^2 + 1 = y^2$, the values of k are :

   Cakravāla method   1, 6, 7, 6, 1

   Euler method   1, 9, 6, 7, 7, 6, 9, 1

   As is evident, both are palindromes.

2. *Cakravāla* represents a shortest possible continued fraction algorithm, with minimum number of cycles. (Naturally, use of *bhāvanā* effects shortening of the calculations).

3. The *cakravāla* always produces the least (positive) solution ; from thereon, all solutions. With or without use of *bhāvanā*, the least solution is obtained.

4. Selenius further observes[29] that : "All quantities produced in the *cakravāla* process have simple counterparts in the ideal (European) continued fraction process".

5. "*Cakravāla* ingeniously avoids large numbers in the calculations".

4.9. The equation $Nx^2 - 1 = y^2$:

Bhāskara has also dealt with the case $Nx^2 - 1 = y^2$ i.e. when the additive or *kṣepa* is negative unity (−1). He says (*BG.* v. 90) : रूपशुद्धौ खिलोद्दिष्टं वर्गयोगो गुणो न चेत् । – When *kṣepa* is negative unity, the solution of the problem is impossible unless the *prakṛti* is the sum of the squares of two rational numbers.

Kṛṣṇa justifies his assertion as follows :

Let the equation be $Nx^2 - 1 = y^2$. Then

$$Nx^2 = y^2 + 1 \text{ , dividing by } x^2$$

$$N = \frac{y^2}{x^2} + \frac{1}{x^2} = \left(\frac{y}{x}\right)^2 + \left(\frac{1}{x}\right)^2$$

which is the sum of squares of two rational numbers.

Therefore Kṛṣṇa says (*BP.* p. 146): यदि प्रकृतिर्वर्गयोगरूपा न भवेत् तर्हि रूपशुद्धौ उद्दिष्टं खिलं ज्ञेयम् । कस्यापि वर्गस्तया प्रकृत्या गुणितो रूपोनः सन्मूलदो नैव भवेदित्यर्थः । – If the *prakṛti* is not a sum of two squares then there cannot be an integral solution for the equation.

4.9.1. European Method of Solving $Nx^2 - 1 = y^2$:

It is not clear whether the European mathematicians who later developed Pell's equation were aware of the above. However, the following discussion[30] seems to hint at the result.

29. C.O. Selenius, "Rationale of the Chakravala process of Jayadeva and Bhaskara II", *op.cit.*, pp. 177-78.

30. *Cf.* C.D. Olds, *op.cit.*, pp. 115-17.

According to a well known result relating to convergents (Appendix III) $y_n^2 - Nx_n^2 = (-1)^n$, when n is the period of the continued fraction for $\sqrt{N}$.

Case 1 : n is even : Then equation $y_n^2 - Nx_n^2 = (-1)^n$ becomes

$$y_n^2 - Nx_n^2 = 1$$

This means that $x = x_n$, $y = y_n$ is a particular solution of $Nx^2 + 1 = y^2$.

Case 2 : n is odd : The equation $y_n^2 - Nx_n^2 = (-1)^n$ becomes

$$y_n^2 - Nx_n^2 = -1$$

showing that $x = x_n$, $y = y_n$ is a solution of $Nx^2 - 1 = y^2$. We have to move to the next period and reach the convergent $\frac{y_{2n}}{x_{2n}}$ to solve $Nx^2 + 1 = y^2$, i.e. $x = x_{2n}$, $y = y_{2n}$ are solutions are $Nx^2 + 1 = y^2$ since $y_{2n}^2 - Nx_{2n}^2 = (-1)^{2n} = +1$.

This shows that particular solutions for $Nx^2 + 1 = y^2$ can always be found. But particular solutions for $Nx^2 - 1 = y^2$ *viz.*, are found only when the period of the continued fraction for $\sqrt{N}$ is odd. Also, not all equations of this type can be solved.

In this connection, we have the following known results[31] on existence and non-existence of solutions for $Nx^2 - 1 = y^2$.

Result 1 : If $N - 3$ is an integral multiple of 4, i.e. $N = 4k + 3$ for some natural number k, the equation $Nx^2 - 1 = y^2$ has no solutions.

Result 2 : If N is a prime number of the form $4k + 1$, then the equation $Nx^2 - 1 = y^2$ always has solutions.

Result 2 is closely connected with a famous theorem stated by Fermat in 1640, and proved by Euler in 1754.

31. *ibid.*, p. 132.

Theorem : Every prime 'p' of the form $4k+1$ can be expressed as the sum of two squares, and the representation is unique. That is, there exists one and only one pair of integers P, Q such that $p = P^2 + Q^2$.

The above two statements conform to what Bhāskara had said five hundred years earlier.

4.9.2. Method for solving $Nx^2 - 1 = y^2$, when N is the sum of two squares :

Bhāskara says (*BG.* v. 91) :

अखिले कृतिमूलाभ्यां द्विधा रूपं विभाजितम् ।
द्विधा ह्रस्वपदं ज्येष्ठं ततो रूपविशोधने ।
पूर्ववद्वा प्रसाध्येते पदे रूपविशोधने ॥

– In case the solution is possible, taking the roots of the two squares that form the equation and dividing 1 (unity) by these roots, the two separate values for x are obtained and from these the corresponding values for y.

4.9.2.1. Theorem : For the equation, $Nx^2 - 1 = y^2$ where $N = m^2 + n^2$ $m, n \neq 0$ a solution is

$$x = \frac{1}{m}\ ,\ y = \frac{n}{m} \qquad \text{or} \qquad x = \frac{1}{n}\ ,\ y = \frac{m}{n}$$

Remarks : This result shows that Bhāskara and Kṛṣṇa were aware of the possibility of rational solutions since, both $\frac{n}{m}$ and $\frac{m}{n}$ cannot be integers unless $n = m = 1$.

4.9.2.2. Kṛṣṇa's Proof : If $x = \frac{1}{m}$, $y = \frac{n}{m}$, then

$$y^2 = \frac{n^2}{m^2} = \frac{N - m^2}{m^2}$$

$$= \frac{N}{m^2} - 1 = Nx^2 - 1$$

Similarly $x = \frac{1}{n}$, $y = \frac{m}{n}$ yields $y^2 = \frac{m^2}{n^2} = \frac{N - n^2}{n^2} = Nx^2 - 1$

If we have $N = m^2 + n^2$ and take the more general equation $Nx^2 - k^2 = y^2$ then

$$x = \frac{k}{m} \;,\; y = \frac{kn}{m} \;;\; x = \frac{k}{n} \;,\; y = \frac{km}{n} \quad \text{are solutions.}$$

Bhāskara explains the above *sūtra* (*BG.* v. 92) with two examples. They are :

त्रयोदशगुणो वर्गः निरेकः कः कृतिर्भवेत् ।
को वाऽष्टगुणितो वर्गः निरेको मूलदो वद ॥

– find the numbers such that – 1) the square of the number multiplied by 13, and unity subtracted from it yields a square ; 2) the square of the number multiplied by 8 and unity subtracted from it yields a square.

To solve for x and y in

1) $13x^2 - 1 = y^2$

2) $8x^2 - 1 = y^2$

Note : Here it is naturally assumed that integral roots are meant since all along Bhāskara has been talking about *abhinna mūlam* (integral roots), in spite of the remark following theorem in **4.9.2.1.**

Example 1 : Solve $13x^2 - 1 = y^2$

13 can be written as $2^2 + 3^2$ or $3^2 + 2^2$. Taking $m = 3$ $n = 2$ the roots by the theorem are $\frac{1}{2}$, $\frac{3}{2}$; taking $m = 2$ $n = 3$ the roots are $\frac{1}{3}$, $\frac{2}{3}$. But these roots are still *bhinna* or fractional. By employing the ingenious method of *bhāvanā* we are able to arrive at the integral solutions (5, 18) since $13(5)^2 - 1 = 18^2$. Taking

$$x = \frac{1}{2} \quad y = \frac{3}{2}$$

$$13\left(\frac{1}{2}\right)^2 - 1 = \left(\frac{3}{2}\right)^2$$

since *kṣepa* is (-1), *tulya-bhāvanā* can be used to get integer solutions.

	kaniṣṭha	*jyeṣṭha*	*kṣepa*
13	$\frac{1}{2}$	$\frac{3}{2}$	-1
	$\frac{1}{2}$	$\frac{3}{2}$	-1

$$x = \frac{1}{2} \times \frac{3}{2} \times 2 = \frac{3}{2}$$

$$y = 13 \times \frac{1}{2} \times \frac{1}{2} + \frac{3}{2} \times \frac{3}{2} = \frac{13}{4} \times \frac{9}{4} = \frac{22}{4} = \frac{11}{2}$$

$$k = -1 \times -1 = +1$$

Since $k = +1$, *samāsa-bhāvanā* has to be done to make $k = -1$

	kaniṣṭha	*jyeṣṭha*	*kṣepa*
13	$\frac{1}{2}$	$\frac{3}{2}$	-1
	$\frac{3}{2}$	$\frac{11}{2}$	$+1$

$$x = \frac{1}{2} \times \frac{11}{2} + \frac{3}{2} \times \frac{3}{2} = \frac{11}{4} + \frac{9}{4} = \frac{20}{4} = 5$$

$$y = 13 \times \frac{1}{2} \times \frac{3}{2} + \frac{3}{2} \times \frac{11}{2} = \frac{39}{4} + \frac{33}{4} = \frac{72}{4} = 18$$

$$k = -1 \times +1 = -1$$

Now, $13(5)^2 - 1 = 18^2$, i.e. $x = 5$, $y = 18$ is a solution of above equation.[32]

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32. The equation $13x^2 - 1 = y^2$ can be solved easily by *cakravāla* also. The auxiliary equation is $13(1)^2 + 3 = 4^2$ and the *kuṭṭaka is* $x + 4 = 3y$ for which solution sets are (2,2), (5,3), (8,4)..., choosing (2,2) since $\frac{13-4}{3}$ is least, $13(2)^2 - 3 = 7^2$.
The next *kuṭṭaka* is $2x + 7 = 3y$, with solution sets (1,3), (4,5), (7,7)..., choosing (4,5) $13(5)^2 - 1 = 18^2$ getting the desired solution.

Remarks : Kṛṣṇa concludes his explanation of the example saying (*BP.* p. 149) : एवं रूपशुद्धौ जाते मूले अभिन्ने । – we have thus obtained integral roots for subtractive unity. He also corroborates Bhāskara's words (*BG.* p. 41) : अत्र सर्वत्र रूपक्षेपजपदाभ्यां भावनया पदानामानन्त्यं ज्ञेयम् । – In all cases like this, an infinite number of roots can be derived by *bhāvanā* with the roots for the additive unity.

If N satisfies the conditions for equations with subtractive unity, then the least positive roots may be rational ; but integral roots will be reached by using *bhāvanā* and / or *cakravāla.*

<u>Example 2 :</u> Solve $8x^2 - 1 = y^2$

Here $N = 8 = 2^2 + 2^2$ i.e. sum of 2 squares (same squares). $y = 1$, $x = \frac{1}{2}$ is a solution set as it is verified $8\left(\frac{1}{2}\right)^2 - 1 = 1^2$. To make the root x integral, *bhāvanā* may be employed.

	kaniṣṭha	*jyeṣṭha*	*kṣepa*
8	$\frac{1}{2}$	1	– 1
	$\frac{1}{2}$	1	– 1

$$x = \frac{1}{2} + \frac{1}{2} = 1$$

$$y = 8 \times \frac{1}{2} \times \frac{1}{2} + 1 = 3$$

$$k = -1 \times -1 = 1$$

By *bhāvanā* again, since the new *kṣepa* is +1,

	kaniṣṭha	*jyeṣṭha*	*kṣepa*
8	$\frac{1}{2}$	1	– 1
	1	3	1

$$x = \frac{1}{2} \times 3 + 1 = \frac{3}{2} + 1 = \frac{5}{2}$$

$$y = 8 \times \frac{1}{2} + 1 \times 3 = 7$$

$$k = -1 \times +1 = -1$$

Bhāvanā has to be done again to get integral solutions

	kaniṣṭha	*jyeṣṭha*	*kṣepa*
8	1	3	1
	$\frac{5}{2}$	7	-1

$$x = 1 \times \frac{5}{2} + 3 \times 7 = \frac{5}{2} + 21 = \frac{47}{2}$$
$$y = 8 \times \frac{5}{2} + 3 \times 7 = 41$$
$$k = -1$$

It can be seen that the solution for x at any stage is not integral though rational.

Remarks : 1) Bhāskara himself does not work out this problem ; he only says (*BG.* p. 41) : एवं द्वितीयोदाहरणे प्रकृतिः ८ प्राग्वज्जाते ह्रस्वज्येष्ठपदे क $\frac{१}{२}$ ज्ये १ । – Eight is the *prakṛti* in the second example ; worked as before we get *kaniṣṭha* root as $\frac{1}{2}$ and *jyeṣṭha* root as 1.

Kṛṣṇa adds, however, in his commentary (*BP.* p. 149) : अथ द्वितीयोदाहरणे प्रकृतिः ८ अयं द्विकयोर्वर्गयोगः प्राग्वज्जाते ह्रस्वज्येष्ठे क $\frac{१}{२}$ ज्ये १, क्षे १ं प्राग्वत् चक्रवालेन अभिन्ने कार्ये । – In the second example *prakṛti* is 8 ; this is a sum of two squares ; therefore working as before the (lesser) *kaniṣṭha* root x is $\frac{1}{2}$ and (greater) *jyeṣṭha* root y is 1 ; *kṣepa* is -1. Employing *cakravāla* as before, integral roots may be derived.

2) While elucidating the rules of *cakravāla*, Bhāskara clearly states (*BG.* v. 88ab) : चतुर्द्व्यैकयुतावेवमभिन्ने भवत: पदे – "thus are integral roots found with four, two or one for additive". [33]

Kṛṣṇa the commentator, who scrupulously follows Bhāskara's words has added the phrase : अभिन्ने कार्ये । – "to make integral". One wonders why this example is given here when it does not produce integral roots. Traditional mathematicians like Sudhakara Dvivedi, Achyuthananda Jha, Jivanatha Jha are either strangely silent on this issue or they quote Bhāskara *verbatim*. Datta and Singh[34] carefully employ the word 'rational' instead of 'integral', in this context.

We establish the following theorem which clinches the issue:

Theorem : The equation $8x^2 - 1 = y^2$ has no integral solutions.

Proof : Let, $8x^2 - 1 = y^2$ where x and y are both integers. Add (-1) to both sides. Then

$$\begin{aligned} 8x^2 - 1 - 1 &= y^2 - 1 \\ 8x^2 - 2 &= y^2 - 1 \\ 2(4x^2 - 1) &= y^2 - 1 \\ 2(2x+1)(2x-1) &= (y+1)(y-1) \qquad ① \end{aligned}$$

Now if x and y are both integers, since the left hand side is a multiple of 2 it is always even. For the right hand side (r.h.s.) also to be even, a) both $(y+1)$ and $(y-1)$ are even in which case the product will be even, b) one of $(y+1)$ and $(y-1)$ is even making the product even.

Case 1 : Both $(y+1)$ and $(y-1)$ are even. Note that if one of $(y+1)$

33. Tr. by H.T. Colebrooke, *op.cit.*, p. 175.

34. Datta and Singh, *op.cit.*, Vol. II, p. 178.

or $(y-1)$ is even so is the other. This is possible only if y is odd. Let y be $= 2m+1$ an odd number. Then

$$\text{r.h.s. of ①} = (2m+1+1)(2m+1-1)$$
$$= (2m+2)(2m) = 4(m+1)(m)$$

Equating both sides we have,

$$2(2x+1)(2x-1) = 4m(m+1)$$

Dividing by 2 throughout,

$$(2x+1)(2x-1) = 2m(m+1) \qquad ②$$

Now the l.h.s. of ② is the product of two odd numbers and hence an odd number. But r.h.s. is a multiple of 2 and hence an even number. There is a contradiction. So our assumption that $(y+1)$ and $(y-1)$ are both even is wrong.

Case 2 : Both $(y+1)$ and $(y-1)$ are odd. Note that if one of $(y+1)$ or $(y-1)$ is odd so is the other. In this case r.h.s. of ① is odd, while the l.h.s. is even, a contradiction. So case 2 is also ruled out.

Therefore the equation $8x^2 - 1 = y^2$ cannot have integral roots.

Note : 1) More generally, if $N=2n^2$ for an even n, then the equation $2n^2x^2 - 1 = y^2$ cannot be solved in integers.

2) It is observed that the general equation $Nx^2 - k^2 = y^2$ (see **4.9.2.2**) cannot be solved unless N is the sum of the squares of two rational numbers (*BG.* v. 90) : रूपशुद्धौ खिलोद्दिष्टं वर्गयोगो गुणो न चेत् ।।

The equation cannot be solved for x , y if the multiplier N is not the sum of squares. Kṛṣṇa gives the very simple argument therefor. Suppose the contrary i.e. N is not the sum of the squares of rationals and $x \neq 0$, necessarily,

then $N = \dfrac{y^2 + k^2}{x^2} = \left(\dfrac{y}{x}\right)^2 + \left(\dfrac{k}{x}\right)^2$, i.e. a sum of squares of two rational numbers, a contradiction to our assumption thereby establishing the observation.

4.10. Variations of the *varga-prakṛti* :

So far, Bhāskara has dealt with *cakravāla* method to arrive at integral solutions. In a few more forms of *varga-prakṛti*, Bhāskara does not insist on integral solutions, and hence does not apply *cakravāla* method. These forms[35] are discussed below as explained by Kṛṣṇa in detail.

Form 1 : (*BG.* v. 93) : वर्गच्छिन्ने गुणे ह्रस्वं तत्पदेन विभाजयेत् ॥ – If the multiplier is divisible by a square, the lesser root is that divided by its root. That is $Mn^2x^2 \pm k = y^2$.

Chinna (छिन्न) here means divisible. Kṛṣṇa explains (*BP.* p. 150) : येन वर्गेण प्रकृतेरपवर्तः कृतः तस्य पदेन कनिष्ठं भाज्यम् । ज्येष्ठं तु यथास्थितमेव । – if the *prakṛti* is divisible by a square i.e. $N = Mn^2$ then the lesser, *kaniṣṭha* root will be that divided by the square root ; the *jyeṣṭha* root remains as it is.

Kṛṣṇa explains this as follows (*BP.* p. 150) : न हि प्रकृतिकृतः विशेषः कनिष्ठेऽस्ति प्रकृतेर्गुणने भजने वा कनिष्ठं गुणितमपवर्तितं वा स्यात् । अतस्तन्मूलेन कनिष्ठमेव भाज्यम् । Kṛṣṇa means that by multiplying and dividing N by n^2, the equation does not change.

$$\therefore \qquad M\frac{x^2}{n^2}n^2 \pm k = y^2$$

$$Mn^2\left(\frac{x}{n}\right)^2 \pm k = y^2$$

so that $\dfrac{x}{n}$, y is a solution for $Mn^2x^2 \pm k = y^2$, if (x, y) is a solution for $Mx^2 \pm k = y^2$.

35. Forms I and II have been dealt with by Brahmagupta also, *vide*, *Br.Sp.* XVIII.70, 69 respectively.

Note : In the context of the last note in **4.9** the observation applies even when $Mn^2x^2 \pm k = y^2$ does not have integral solutions.

Form 2 : $a^2x^2 \pm k = y^2$

Bhāskara takes up the equation of the type $a^2x^2 \pm k = y^2$. i.e. where the *prakṛti* is a square. In Kṛṣṇa's words (*BP.* p. 151) : प्रकृतौ वर्गरूपाणां पदानयने । . . . अत्र गुणमूलहृतस्त्वाद्य इत्युक्तेर्यत्र वर्गरूपा प्रकृतिर्भवति तत्रैवास्य सूत्रस्यावसर इति ज्ञेयम् । – i.e. a square *prakṛti* is here considered.

Bhāskara's rule in the case of $a^2x^2 + k = y^2$ is as follows (*BG.* v. 97) :

इष्टभक्तो द्विधा क्षेप इष्टोनाढ्यो दलीकृतः ।
गुणमूलहृतश्चाऽऽद्यो ह्रस्वज्येष्ठे क्रमात्पदे ॥

– The *kṣepa* divided by an optional number is set down at two places ; the quotient is diminished at one place and increased at the other by that optional number and then halved. The former is again divided by the square root of the *prakṛti*. The new quotients thus arrived at are respectively the *kaniṣṭha* and *jyeṣṭha* roots.

Note : If m is the optional number the rule is just a re-statement of the identity $\frac{1}{a^2}\,\frac{a^2}{4}\left(\frac{k}{m} - m\right)^2 + k = \frac{1}{4}\left(\frac{k}{m} + m\right)^2$ derived from $(a - b)^2 + 4\,ab = (a + b)^2$.

The rationale of the above solution is treated substantially by Kṛṣṇa.

Upapatti **1** : Let the equation be $Nx^2 \pm k = y^2$ where N is a square namely a^2. i.e. the equation is $a^2x^2 + k = y^2$. At the outset Kṛṣṇa says (*BP.* p. 151) : यत्र वर्गरूपा प्रकृतिस्तत्र क्षेपाभाव एव ज्येष्ठमूलं लभ्यते । यतः कनिष्ठवर्गे वर्गरूपप्रकृत्या गुणिते वर्ग एव स्यात् । – If the *kṣepa* is not there i.e. if *kṣepa* = 0 , then $a^2x^2 = y^2$. y is known immediately ; i.e. $y = ax$ (considering only the positive root).

Now if we add a *kṣepa* to it, (*BP.* p. 151): यतोऽस्य मूलं प्रथमज्येष्ठात् किंचित् अधिकं स्यात् । . . . अत्र किमधिकमिति न ज्ञायते । – the new 'y' or greater root will be greater (in value) than the original 'y' . But we do not know how much greater. So Kṛṣṇa assumes some value for *kṣepa* 'k' – तदिष्टं प्रकल्प्य जात: क्षेप: । Let k be assumed to be $2my + m^2$ where m is any arbitrary rational number.

$$k = 2my + m^2 \ , \ \frac{k}{m} = 2y + m \text{ , applying the rule "इष्ट आढ्य:"}$$

$$\frac{k}{m} + m = 2y + m + m = 2y + 2m$$

$$\text{thus} \quad \frac{1}{2}\left(\frac{k}{m} + m\right) = \frac{1}{2}(2y + 2m) = y + m$$

i.e. new greater root $\frac{1}{2}\left(\frac{k}{m} + m\right)$ for any arbitrary number m,

Kaniṣṭha root is $\frac{'y'-m}{a} = \frac{1}{2}\frac{\left\{\left(\frac{k}{m}+m\right)-m\right\}}{a} = \frac{1}{2a}\left(\frac{k}{m}-m\right)$ as indicated in the Note preceding ***Upapatti* 1.**

In Kṛṣṇa's words the above derivation is (*BP.* p. 152) : तथा च इष्ट भक्तो द्विधा क्षेप इष्टोनाढ्यो दलीकृत इत्यनेन केवलज्येष्ठमिष्टाधिकज्येष्ठं च साधितम् । तत्र केवलज्येष्ठं गुणमूलभक्तं सत्कनिष्ठं भवतीत्यत उक्तं गुणमूलहृतश्चाद्य इति । – [36]Thus the *kṣepa* divided

36. The above can be written differently as below :

If the r.h.s. $-k$ were to be a square for a solution y let it be $(y - m)^2$ for some m.

i.e. $y^2 - k = (y - m)^2 = y^2 - 2my + m^2$. So, $-k = -2my + m^2$ or $-\frac{k}{m} = m - 2y$,

i.e. $2y = m + \frac{k}{m}$, $y = \frac{1}{2}\left(m + \frac{k}{m}\right)$. Substituting for y in equation $a^2x^2 + k = y^2$

$$a^2x^2 + k = \left[\frac{1}{2}\left(\frac{k}{m} + m\right)\right]^2 = \frac{1}{4}\frac{k^2}{m^2} + \frac{k}{2} + \frac{m^2}{4} \quad \text{or}$$

$$a^2x^2 = \frac{1}{4}\frac{k^2}{m^2} - \frac{k}{2} + \frac{m^2}{4} = \left[\frac{1}{2}\left(\frac{k}{m} - m\right)\right]^2$$

$$\text{so that} \quad x = \frac{1}{2a}\left(\frac{k}{m} - m\right)$$

by the optional number m is decreased by the optional number and then divided by $2a$ to get the lesser root $\frac{y}{a}$. The *kṣepa* divided by the optional number m is increased by the optional number and then halved to get the greater root $y + m$.

***Upapatti* 2 :** Kṛṣṇa gives a second proof which is followed by most of the later authors.

$$a^2x^2 + k = y^2$$

implies $k = y^2 - a^2x^2$

$$= (y - ax)(y + ax)$$

Assume $y = ax + m$, m being any arbitrary rational number.

$$y - ax = m$$

i.e. $$k = m(y + ax)$$

i.e. $$\frac{k}{m} = y + ax \quad ①$$

$$m = y - ax \quad ②$$

Adding ① and ② $\frac{k}{m} + m = 2y$

$$y = \frac{1}{2}\left(\frac{k}{m} + m\right)$$

subtracting ② from ① $\frac{k}{m} - m = 2ax$

$$x = \left(\frac{k}{m} - m\right)\frac{1}{2a}$$

The above steps are quite naturally described by Kṛṣṇa (*BP.* p. 152) :

<u>Step I :</u> वर्गरूपप्रकृत्यागुणितकनिष्टवर्गो वर्ग एव । अथ क्षेपेऽपि क्षिप्ते यदि वर्गः स्यात् तर्हि क्षेपो वर्गान्तरमेव । – x^2 multiplied by another square a^2 the resultant

product a^2x^2 which is also a square. If *kṣepa* k is added to a^2x^2 then clearly $k = y^2 - a^2x^2$. i.e. *kṣepa* is the difference of squares.

Step 2 : अथ वर्गान्तरं राशिवियोगभक्तं योग: तत: प्रोक्तवदेव राशी इत्युक्तत्वादत्रान्तरमिष्टं कल्प्यते । – The difference between the two squares divided by the difference of the square roots is equal to the sum of the square roots. Therefore the difference of the square roots $(y - ax)$ is assumed to be any arbitrary rational number m.

Step 3 : तेन क्षेपरूपे वर्गान्तरे भक्ते योगो लभ्यते । तत: संक्रमणसूत्रेण राशीज्ञानं सुलभम् । – That *kṣepa* divided by $(y - ax)$ gives $(y + ax)$. Solving the two equations by the rule of *saṅkramaṇa* (concurrence : simultaneous linear equation)[37] x and y can be obtained easily.

अनयैव युक्त्या ऋणक्षेपेऽपि बोध्यम् । – The same method can be used for $(-k)$ also.

At this juncture Kṛṣṇa shows concern about the terminology 'greater and smaller' and makes the following observation : धनक्षेपे बृहद्राशिरुद्दिष्टज्येष्ठं ऋणक्षेपे तु लघुराशिरुद्दिष्टज्येष्ठम् । – If the *kṣepa* is positive then the 'greater root y' will be greater than x ; if the *kṣepa* is negative the 'greater root y' will be smaller than 'x'. Therefore in the case of subtractive *kṣepa* $(-k)$ the reading should be इष्टाढ्योनो दलीकृत instead of इष्टोनाढ्यो दलीकृत: i.e. $y = \frac{1}{2}\left(\frac{k}{m} - m\right)$ and $x = \frac{1}{2a}\left(\frac{k}{m} + m\right)$. यद्यपि क्षेपस्य ऋणत्वाङ्कनेन यथाश्रुत एव पाठे अयमर्थ: संपद्यते तथापि कनिष्ठज्येष्ठयो: ऋणत्वं स्यात् तस्मादृणत्वाङ्कनं विनैव इष्टाढ्योन इति पाठव्यत्ययेन पदसाधनमृणक्षेपे द्रष्टव्यम् । But if the original reading is to be retained then both x and y should be negative. So instead of making both x and y negative, *iṣṭonāḍhyo* could be changed to *iṣṭāḍhyono* and the roots can be obtained. Kṛṣṇa's observation can be mathematically explained as follows :

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37. See **5.4** for *Saṅkramaṇa*

If in the equation $a^2x^2 + k = y^2$, $k = -k'$, $k' > 0$.

then $a^2x^2 - k' = y^2$

$$-k' = y^2 - a^2x^2 = (y - ax)(y + ax)$$

Assume $(y - ax)$ to be m where m is any artibitary rational number. Then

$$-k' = m(y + ax)$$

$$\frac{-k'}{m} = y + ax \qquad ①$$

Also by assumption, $m = y - ax$ ②

Adding ① and ② $\frac{-k'}{m} + m = 2y$

$$y = \frac{1}{2}\left(\frac{-k'}{m} + m\right)$$

Subtracting ② from ① $\frac{-k'}{m} - m = 2ax$

$$x = \frac{1}{2a}\left(\frac{-k'}{m} + m\right)$$

If we take both x and y to be negative, say $x' = -x$, $y' = -y$,

$$y' = -y = -\frac{1}{2}\left(\frac{-k'}{m} + m\right) \quad \text{– इष्टोनदलीकृत}$$

$$= \frac{1}{2}\left(\frac{k'}{m} - m\right)$$

$$x' = -x = \frac{-1}{2a}\left(\frac{-k'}{m} - m\right) \quad \text{– इष्टाढ्यदलीकृत}$$

$$= \frac{1}{2a}\left(\frac{k'}{m} + m\right)$$

Therefore it can be said that y is the root obtained by इष्टोनदलीकृतं and x is the root obtained by इष्टाढ्यदलीकृतगुणमूलहृतं ।

<u>Form 3</u> : $Nx^2 \pm N = y^2$ i.e. (*BP.* p. 153) : प्रकृतिसमक्षेप – *prakṛti* and *-pa* are the same. The two examples given are :

$$13x^2 - 13 = y^2 \qquad ①$$
$$13x^2 + 13 = y^2 \qquad ②$$

Bhāskara solved the examples. However it is Kṛṣṇa who gives the method for both of them. Take equation ① . Then

$$13(1)^2 - 13 = 0$$

so that, $x = 1$, $y = 0$ or $(1,0)$ is a solution set for the equation ① (*BP.* p. 153) : अथ ज्येष्ठस्य शून्यत्वे यदि लोकस्य प्रतीतिर्नास्ति तर्हि रूपक्षेपपदोथ्थया भावनया आनन्त्यमिति ज्ञापयितुमाह । – If we cannot accept a solution where $y = 0$ then we have to resort to *bhāvanā* with the equation where *kṣepa* is a positive one. We will then have infinite solutions. To see what is suggested above, consider the equation $13x^2 + 1 = y^2$. Using the sutra (*BG.* v. 83) : इष्टवर्गप्रकृत्योर्यद्विवरं...। assume $m = 3$,

$$\text{then} \quad x = \frac{2m}{N - m^2} = \frac{2 \times 3}{13 - 9} = \frac{6}{4} = \frac{3}{2}$$

$$13(x)^2 + 1 = y^2 = 13\left(\frac{3}{2}\right)^2 + 1 = 13 \times \frac{9}{4} + 1 = \frac{121}{4}$$

So $y = \sqrt{\dfrac{121}{4}} = \dfrac{11}{2}$, i.e. $\left(\dfrac{3}{2}, \dfrac{11}{2}\right)$ is a solution set of $13x^2+1 = y^2$. Using *bhāvanā* between solution sets $(1,0)$ $\left(\dfrac{3}{2}, \dfrac{11}{2}\right)$ we have

prakṛti	*kaniṣṭha*	*jyeṣṭha*	*kṣepa*
13	1	0	−13
	$\frac{3}{2}$	$\frac{11}{2}$	1

$$\text{New } x(x_1) = 1 \times \frac{11}{2} + 0 \times \frac{3}{2} = \frac{11}{2}$$

$$\text{New } y(y_1) = 13 \times 1 \times \frac{3}{2} + 0 \times \frac{11}{2} = \frac{39}{2}$$

New *kṣepa* (k_1) $= -13 \times 1 = -13$

i.e. $$13\left(\frac{11}{2}\right)^2 - 13 = \left(\frac{39}{2}\right)^2$$

So $\left(\frac{11}{2}, \frac{39}{2}\right)$ is a solution set of the equation $13x^2 - 13 = y^2$.

From the *sūtra* (*BG.* v. 91) : अखिले कृतिमूलाभ्यां...। , the least positive solution for $13x^2-1=y^2$ is $13\left(\frac{1}{2}\right)^2 - 1 = \left(\frac{3}{2}\right)^2$, i.e. solution set is $\left(\frac{1}{2}, \frac{3}{2}\right)$ for $13x^2-1=y^2$. Kṛṣṇa uses the word विशेषसमभावनया and explains it saying विशेषभावना अन्तरभावना । समभावना समासभावना । Using *samāsa-bhāvanā* between solution sets $\left(\frac{11}{2}, \frac{39}{2}\right)$ and $\left(\frac{1}{2}, \frac{3}{2}\right)$ we have

prakṛti	*kaniṣṭha*	*jyeṣṭha*	*kṣepa*
13	$\frac{11}{2}$	$\frac{39}{2}$	-13
	$\frac{1}{2}$	$\frac{3}{2}$	-1

New $x(x_2)$ $= \frac{11}{2} \times \frac{3}{2} + \frac{39}{2} \times \frac{1}{2} = \frac{33}{4} + \frac{39}{4} = \frac{72}{4} = 18$

New $y(y_2)$ $= 13 \times \frac{11}{2} \times \frac{1}{2} + \frac{39}{2} \times \frac{3}{2} = \frac{143}{4} + \frac{117}{4} = \frac{260}{4} = 65$

New *kṣepa* k_2 $= -13 \times -1 = 13$

So $$13(18)^2 + 13 = 65^2$$

Thus $(18, 65)$ is a solution set of $13x^2 + 13 = y^2$.

Using *antara-bhāvanā* between solution sets $\left(\frac{11}{2}, \frac{39}{2}\right)$ and $\left(\frac{1}{2}, \frac{3}{2}\right)$ we have

prakṛti	*kaniṣṭha*	*jyeṣṭha*	*kṣepa*
13	$\frac{1}{2}$	$\frac{3}{2}$	-1
	$\frac{11}{2}$	$\frac{39}{2}$	-13

$$\text{New } x(x_3) = \frac{1}{2} \times \frac{39}{2} - \frac{11}{2} \times \frac{3}{2} = \frac{6}{4} = \frac{3}{2}$$

$$\text{New } y(y_3) = 13 \times \frac{1}{2} \times \frac{11}{2} - \frac{3}{2} \times \frac{39}{2} = \frac{143 - 117}{4} = \frac{26}{4} = \frac{13}{2}$$

$$\textit{kṣepa} = -1 \times -13 = +13$$

Thus $\left(\frac{3}{2}, \frac{13}{2}\right)$ is also a solution for $13x^2 + 13 = y^2$. To verify $13\left(\frac{3}{2}\right)^2 + 13 = \left(\frac{13}{2}\right)^2$.

Form 4 : The last type Bhāskara discusses is (*BG.* v. 100) : $-Nx^2 + k = y^2$. "The equation of the form $c - Nx^2 = y^2$ has not been considered by any Hindu algebraist as deserving of special treatment"[38]. It occurs, however, incidentally in examples. Bhāskara has given the example $-5x^2 + 21 = y^2$. Kṛṣṇa first notes that $x = 1$, $y = 4$ is a solution : i.e. $-5(1)^2 + 21 = 16$. Also that $x = 2$, $y = 1$ is a solution, i.e. $-5(2)^2 + 21 = 1$

He suggests (*BP.* p. 154) : रूपक्षेप भावनया पदानन्त्यं प्राग्वत् । – As before, an infinite number of solutions can be obtained by *bhāvanā* for this equation.

4.11. *Cakravāla* in the modern context :

The *cakravāla* method discussed above, has pre-empted the European methods by nearly a thousand years. Hankel,[39] the German

38. Datta and Singh, *op.cit.*, Vol. II, pp. 177-78.

39. A.A.K. Ayyangar, "New Light on Bhaskara's Chakravala or Cyclic Method of solving Indeterminate Equations of the Second Degree in two Variables", *op.cit.*, p. 248.

mathematician praises the method of *cakravāla*, and says : " . . . it is certainly the finest thing achieved in the theory of numbers before Lagrange''. Referring to this cyclic method T.L. Heath[40] observes : ''If the Greeks did not accomplish the general solution of our equation, it is all the more extraordinary we should have such a general solution in practical use among the Indians . . . ''.

To cite a very recent appreciation, Selenius pays the highest compliment when he says ''the old Indian *chakravāla* method for solving the mathematically fundamental indeterminate *varga-prakṛti* equation . . . was a very natural, effective and labour-saving method with deep-seated mathematical properties . . . More than ever are the words of Hankel . . . valid, that the *chakravāla* method was the absolute climax ('ohne Zweifel der Glanzpunkt') of the old Indian science, and so of all Oriental mathematics . . . No European performances in the whole field of algebra at a time much later than Bhāskara's, nay nearly up to our times, equalled the marvellous complexity and ingenuity of *chakravāla*''.[41]

40. *ibid.*, p. 226.

41. C.O. Selenius, ''Rationale of the Chakravala process of Jayadeva and Bhaskara II'', *op.cit.*, p.180

CHAPTER - 5
EKAVARṆA SAMĪKARAṆA – MADHYAMĀHARAṆA

Here we deal with application of *bīja* in relation to single variable. Kṛṣṇa discusses the division into classes of equations. Linear equations in one variable (*ekavarṇa samīkaraṇa*) are taken up first for discussion. Bhāskara provides examples from other branches of mathematics also. *Saṅkramaṇa* or concurrence is explained. It is a special case of simultaneous equations of two variables. It is dealt with here because it is used by Bhāskara and Kṛṣṇa in some of the illustrations. Quadratic equations of single variable (*madhyamāharaṇa*) is the next topic. The rule of Srīdhara to find roots of the quadratic is mentioned. Though every quadratic equation has two roots, the cases where they cannot be accepted are also discussed.

5.1. Classification of Equations :

The early Indian classification of equations is observed to have been according to their degrees, such as simple (technically called *yāvat tāvat*) quadratic (*varga*), cubic (*ghana*) and biquadratic (*varga -varga*).

Brahmagupta was probably the first to classify equations according to degrees. He, in his *Br.Sp.*(XVIII, vv. 43, 53) has divided equations into three categories : 1) equations in one unknown (*ekavarṇa samīkaraṇa*) ; 2) equations in several unknowns (*anekavarṇa samikaraṇa*) and 3) equations involving products of unknowns (*bhāvita*). The first category *ekavarṇa samīkaraṇa* is again divided into 1) linear equations and 2) quadratic equations (*avyakta varga samīkarana*).

The commentator Pṛthūdakasvāmi of *Br.Sp.* has made slight variations. His four classes are : 1) linear equations with one unknown ; 2) linear equations with many unknowns ; 3) equations with one, two or more unknowns in their second and higher powers and 4) equations involving products of unknowns. He called the third subdivision as

madhyamāharaṇa since it was based on the principle of elimination of the middle term.[1] This method of classification has been followed by later writers. Bhāskara (*BG.* p. 44) followed this classification more explicitly according to the following description : 1) प्रथममेकवर्णसमीकरणबीजम् – equations with one unknown ; 2) द्वितीयमनेकवर्णसमीकरणबीजम् – equations with many unknowns ; 3) यत्र वर्णस्य द्वयोर्वा बहूनां वर्गादिगतानां समीकरणं तन्मध्यमाहरणम् – equations in one, two or many unknowns in their second or higher powers and 4) यत्र भावितस्य तद्भावितमिति बीजचतुष्टयं (वदन्त्याचार्याः) – equations involving product of two or more unknowns.

According to Kṛṣṇa (*BP.* pp. 155-56), however, equations are primarily of two kinds. मुख्यो विभागस्तु द्वेधैव – *ekavarṇa samīkaraṇa* and *anekavarṇa samīkaraṇa.* He defines the two as : (1) तत्र समशोधनादिनाऽव्यक्तराशेर्मानमवगन्तुं यत्रैकमेव वर्णमधिकृत्य पक्षयोः साम्यं क्रियते तदेकवर्णसमीकरणम् । – where the equation deals with only one unknown it is called *ekavarṇa samīkaraṇa.* Kṛṣṇa adds that : *Ekavarṇa samīkaraṇa* is again of two kinds – एकवर्णसमीकरणं मध्यमाहरणं चेति । – namely *ekavarṇa samīkaraṇa* (linear equation of one unknown) and *madhyamāharaṇa* [2] (equations of higher powers of one unknown).

(2) यत्तु अनेकान्वर्णानधिकृत्य पक्षसाम्यं क्रियते तदनेकवर्णसमीकरणम् – where the equation deals with many unknowns it is called *anekavarṇa samīkaraṇa.* This again, is of three classes : अनेकवर्णसमीकरणं मध्यमाहरणं भावितं चेति । – *anekavarṇa samīkaraṇa* (linear equation of many unknowns), *madhyamāharaṇa* (equations of higher powers of many unknowns) and *bhāvita*[3] (where the equations deals with product of unknowns).

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1. Datta and Singh, *op.cit.,* Vol. II, p.35.
2. Kṛṣṇa gives the meaning of the term as (*BP.* p. 156) : यतोऽस्मिन्वर्गराशेर्मूलग्रहणे द्वयोरभिहति द्विघ्नो शेषात्त्यजेदित्यनेन मध्यमस्य खण्डस्याहरणमपनयनं प्रायो भवत्यतो मध्यमाहरणमित्युच्यते । – In this type of equations the middle term (मध्यम) is eliminated (आहरण) to arrive at the solution. Hence the name.
3. *loc.cit.* : *Bhāvita* is defined as : यत्र तु भावितमधिकृत्य साम्यं क्रियते तद्भावितमित्युच्यते – where the equations deal with product of unknowns, it is called *bhāvita.*

Thus Kṛṣṇa says : एवं पंचधापि विभागः संभवति । – Thus there are totally five divisions. He adds : अत्र मध्यमाहरणयोस्तत्वेनैकीकरणे चतुर्धापि विभागः । – If the two *madhyamāharaṇa* equations are clubbed together we have four divisions.

Kṛṣṇa also explains in detail why (according to him) there are only two divisions. According to him *madhyamāharaṇa* equations of the second or higher degree of a single variable are still part of *ekavarṇa samīkaraṇa* ; equations of second or higher degree of several variables and *bhāvita* equations are part of *anekavarṇa samīkaraṇa.* If you say 'how can *ekavarṇa samīkaraṇa* and *madhyamāharaṇa* be grouped together when their definitions are contrary to each other', Kṛṣṇa says (*BP.* p. 155) : ननु तथापि विरुद्धधर्माक्रान्तयोः एकानेकवर्णसमीकरणविशेषयोर्विरुद्धधर्मात् प्रापकेन मध्यमाहरणत्वेन कथं क्रोडीकरणमिति चेत् – पृथिवीत्वतेजस्त्वाक्रान्तयोः पार्थिवतैजसशरीरयोः शरीरत्वेनैवावगच्छ । – the quality of earth and the quality of fire can both coexist in the same body though the earthness and lumniscence of earth and fire are totally different from earth[4]. – Therefore Kṛṣṇa asserts that there are only two divisions.

Kṛṣṇa adds (*BP.* p. 156) : श्रीमद्भास्कराचार्याणां तु बीजद्वैविध्यमेवाभिमतमस्तीति लक्ष्यते । यतस्ते प्रथममेकवर्णसमीकरणं बीजं द्वितीयमनेकवर्णसमीकरणं बीजमिति प्रथमद्वितीयशब्दार्थपूर्वकं विभागमविधाय तदनु यत्र वर्गवर्णस्य द्वयोर्बहूनां वा वर्गादिगतानां समीकरणं तन्मध्यमाहरणं यत्र भावितस्य तद्भावितमिति बीजचतुष्टयं वदन्त्याचार्याः इति वक्ष्यन्ति । अत्र हि यत्रेति बीजद्वयमनूद्य मध्यमाहरणत्व भावितत्वविधानप्रतीतेर्मुख्यं द्वैविध्यमेव प्रतीयते । – Bhāskara is basically of the opinion that equations are only of two types because : 'the first kind is called *ekavarṇa samīkaraṇa* and the second is called *anekavarṇa samīkaraṇa* ; 'where' equation of second or higher degree are dealt with, they are called *madhyamāharaṇa* and 'where' products are involved, they are called *bhāvita*'. The use of the word 'where' clearly denotes that divisions are only two; the third and fourth are subdivisions of the first and second only.

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4. Prof. S. Kuppuswami Sastri, *A Primer of Indian Logic*, The K.S.R. Institute, Madras, 1998, p.4.

Interestingly, Kṛṣṇa also quotes the words of Bhāskara at the end of both the chapters and closes the discussion about classification. He says (*BP.* p. 156) : किं च विशेषस्वरूपैकवर्णसमीकरणसमाप्तावित्येकवर्णसमीकरणबीज-मित्यनुक्तैव । मध्यमाहरणविशेषसमाप्तावित्येकवर्णसमीकरणं बीजम् । – that it has not been said at the end of the chapter that ''the chapter of equation of single variable is ended'' But at the end of the chapter on *madhyamāharaṇa* it has been said, ''thus ends the chapter on *ekavarṇa samīkaraṇa*''.

With all the above arguments, Kṛṣṇa comes to the conclusion – तस्मात् मुख्यो विभागस्तु द्वेधैव – therefore, there are only two divisons.

Remarks : From this lengthy discussion on the classification of the equations it could be assumed that, Kṛṣṇa seems to have noticed the need for knowledge about the number of variables, one or more (even as products) in concurrence with the later Western trend about the degree of these equations.

5.2. Definition of Linear Equation with one Unknown :

The three *sūtras* detailing the procedure of solution of a linear equation of a single variable are as follows (*BG.* vv. 102-04) :

यावत्तावत्कल्प्यमव्यक्तराशेर्मानं तस्मिन्कुर्वतोद्दिष्टमेव ।
तुल्यौ पक्षौ साधनीयौ प्रयत्नात्त्यक्त्वा क्षिप्त्वा वाऽपि संगुण्य भक्त्वा ।।
एकाव्यक्तं शोधयेदन्यपक्षाद्रूपाण्यन्यस्येतरस्माच्च पक्षात् ।
शेषाव्यक्ते नोद्धरेद्रूपशेषं व्यक्तं मानं जायतेऽव्यक्तराशेः ।।
अव्यक्तानां द्व्यादिकानामपीह यावत्तावद्द्व्यादिनिघ्नं हृतं वा ।
युक्तोनं वा कल्पयेदात्मबुद्ध्या मानं क्वापि व्यक्तमेवं विदित्वा ।।

– Some symbol, say, x is assumed for the value of the unknown quantity. By addition, subtraction, multiplication and / or division, the equation should be obtained; one should subtract the unknown of one side from the

other side and the absolute numbers from the opposite side. Dividing the (resulting) absolute number (on that side) by the coefficient of the unknown, the value of the unknown is obtained. If the number of unknowns is two or more, one is assumed to be the unknown and symbolised and its value is found after suitable values are given for the other unknowns.

Note : The procedure is quite the same as adopted in modern mathematics. Though not explicitly stated, it applies only to linear equations in one variable. The remark about solving equations in more than one variable can give only particular solutions but not all solutions.

5.3. Methods of solving Equations in one variable :

In this section, by way of examples, Bhāskara covers many branches of mathematics, such as the rule of three (*trairāśika*), arithmetic progression (*śreḍīphala*) and geometry (*kṣetravyavahāra*). In his commentary Kṛṣṇa adds (*BP.* p. 158) : अथ यत्र द्व्यादयो अव्यक्तराशयो भवेयुः तत्र यद्यपि अनेकवर्णसमीकरणेन उदाहरणसिद्धिरस्ति तथापि बुद्धिवैचित्र्यार्थम् अत्राप्याह अव्यक्तानां द्व्यादिकानाम् अपीति । – Where two or more variables are involved it is better to use *anekavarṇa samīkaraṇa*, but in order to test the intellect, the method is given in the verse (*BG* 104) '*avyaktānām dvyādikānām api ...*'

Note : What Kṛṣṇa, following Bhāskara, means is that the case of two or more variables is reduced to that of one, by a clever device.

Example : (*BG.* v. 108) :

एको ब्रवीति मम देहि शतं धनेन त्वत्तो भवामि हि सखे द्विगुणस्ततोन्यः ।
ब्रूते दशार्पयसि चेन्मम षड्गुणोऽहं त्वत्तस्तयोर्वद धने मम किं प्रमाणे ॥

– One says: 'give me a hundred, friend, I shall then become twice as rich as you.' The other replies, 'If you give me ten, I shall become six times as rich as you.' What is the amount of their respective capitals ?

Bhāskara gives two solutions, one using *ekavarṇa samīkaraṇa* and the other later in the chapter on *anekavarṇa samīkaraṇa.* In modern terminology, this is called simultaneous equations in two variables (see the preceding Note). Following Bhāskara, Kṛṣṇa gives another method.

5.3.1. Kṛṣṇa's Method for *Ekavarṇasamīkaraṇa :*

Let x be the wealth of first man and y the wealth of second man. Then

$$x + 100 = 2(y - 100) \qquad ①$$

$$y + 10 = 6(x - 10) \qquad ②$$

To solve the equation, with one unknown, Kṛṣṇa assumes x to be $z + 10$, $y = 6z - 10$ so that the second equation immediately holds. For,

$$6z - 10 + 10 = 6\,(z + 10 - 10)$$

Substituting the assumed values of x, y in terms of z in equation ①.

$$z + 10 + 100 = 2(6z - 10 - 100)$$

$$z + 110 = 12z - 220$$

$$11z = 330$$

$$z = 30$$

So, $$x = z + 10 = 30 + 10 = 40$$

$$y = 6z - 10 = 180 - 10 = 170$$

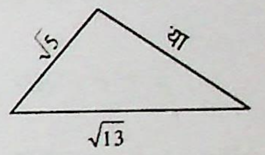
i.e. the wealth of the two are 40 and 170 units respectively.

Remarks : If we express y in terms of x using equation ② we have $y = 6x - 70$ and substituting in ① we have $x + 100 = 2(6x - 70 - 100)$, so that $11x = 100 + 140 + 200$ giving $x = 40$ immediately yielding $y = 6x - 70 = 240 - 70 = 170$.

This is a clever precursor to the modern method of elimination of one variable. *Ekavarṇa samīkaraṇa* is better interpreted as reduction of many variables to one in this context.

Example : Another interesting example is taken from *kṣetravyavahāra* (geometry) (*BG.* v. 119) :

त्रयोदश तथा पञ्च करण्यौ भुजयोर्मिती ।
भूरज्ञाताऽत्र चत्वारः फलं भूमिं वदाऽऽशु मे ॥



– Two sides of a triangle are $\sqrt{13}$ and $\sqrt{5}$. If its area is 4, what is the third side.

Bhāskara assumes the base to be $\sqrt{13}$ and finds out the third side. Kṛṣṇa has a different method assuming base to be the unknown x. On the basis of the *Līlāvati sūtra* (v. 171)[5], the projections of the two sides on the base can be found.

Note : The *sūtra,* incidentally, gives values of the projections of the other two sides of the triangle on the side being considered. In modern notation, the *sūtra* corresponds to the set of formulae

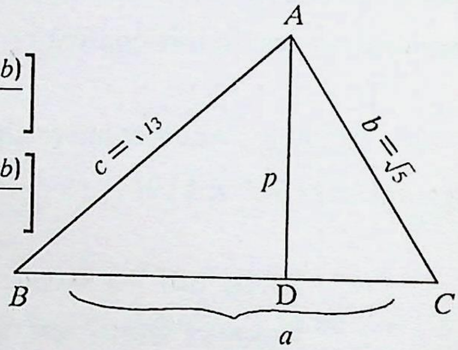
$$a^2 = b^2 + c^2 - 2bc\cos A,\ b^2 = c^2 + a^2 - 2ca\cos B,\ c^2 = a^2 + b^2 - 2ab\cos C.$$

5. त्रिभुजे भुजयोर्योगस्तदन्तरगुणो भुवा हृतो लब्ध्या ।
द्विस्थाभूरूनयुता दलिताऽऽबाधे तयोः स्याताम् ॥

5.3.2. Kṛṣṇa's Method using *Madhyamāharaṇa* :

Kṛṣṇa does not follow Bhāskara here but solves the problem on his own with the use of *madhyamāharaṇa.*

In the figure given, a, b, c are sides of ΔABC and p the altitude. from A on BC. BD and CD are projections of AB and AC respectively on BC



$$\text{Projection } BD = \frac{1}{2}\left[a + \frac{(c+b)(c-b)}{a}\right]$$

$$\text{Projection } CD = \frac{1}{2}\left[a - \frac{(c+b)(c-b)}{a}\right]$$

$$p^2 = c^2 - BD^2 \quad \text{or} \quad b^2 - CD^2$$

$$\text{Also, Area } \Delta ABC = \frac{1}{2}\, p \times a$$

In the given example $c = \sqrt{13}$; $b = \sqrt{5}$; Area = 4 ; p is the altitude ; $a = x$ (unknown). Now

$$BD = \frac{1}{2}\left[a + \frac{(c+b)(c-b)}{a}\right]$$

$$= \frac{1}{2}\left[x + \frac{(\sqrt{13}+\sqrt{5})(\sqrt{13}-\sqrt{5})}{x}\right]$$

$$= \frac{1}{2}\left[x + \frac{(13-5)}{x}\right] = \frac{1}{2}\left[x + \frac{8}{x}\right]$$

$$= \frac{x^2+8}{2x}$$

$$\text{So } BD^2 = \left(\frac{x^2+8}{2x}\right)^2$$

$$= \frac{x^4+16x^2+64}{4x^2}$$

$$c^2 - BD^2 = \left(\sqrt{13}\right)^2 - \frac{x^4 + 16x^2 + 64}{4x^2}$$

$$= \frac{52x^2 - x^4 - 16x^2 - 64}{4x^2} = p^2 \qquad ①$$

p can also be found out in another way (*BP.* p. 171) : अथवा प्रकारान्तरेण लंबगुणं भूम्यर्धं क्षेत्रफलं व्यस्तविधिना भूम्यर्धेन या $\frac{1}{2}$ क्षेत्रफलं भक्तजातो लंब: $\frac{8}{x}$ । – Since Area of a triangle is the product of the attitude and half the base $\left(A = \frac{1}{2}pa\right)$, by reverse rule, the attitude should be the Area divided by half of base $\left(p = \frac{A}{\frac{a}{2}}\right)$. Therefore in the given example we have

$$p = \frac{8}{x} \text{ or } p^2 = \frac{64}{x^2} \qquad ②$$

Equating results ① and ②

$$\frac{64}{x^2} = \frac{52x^2 - x^4 - 16x^2 - 64}{4x^2}$$

i.e. $$\frac{256}{4x^2} = \frac{52x^2 - x^4 - 16x^2 - 64}{4x^2}$$

i.e. $$256 = 52x^2 - x^4 - 16x^2 - 64$$

i.e. $$x^4 - 36x^2 + 320 = 0$$

i.e. $$x^4 - 36x^2 = -320$$

i.e. $$x^4 - 36x^2 + 324 = -320 + 324$$

i.e. $$(x^2 - 18)^2 = 2^2$$

i.e. $$x^2 - 18 = \pm 2, \quad x^2 = 18 \pm 2, \quad x^2 = 20, 16$$

$$x^2 = 20 \quad \text{gives} \quad x = \sqrt{20}$$

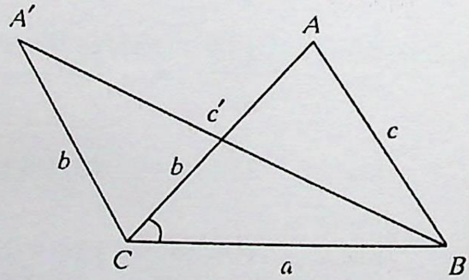
$$x^2 = 16 \quad \text{gives} \quad x = 4$$

It may be noted that the equation reduces to one of solving a quadratic equation in x^2. In solving it by completing the square in the above steps, the process is one of eliminating the x^2 term, in the quadratic in x^4. This is thus a case of *madhyama - āharaṇa.*

Kṛṣṇa states (*BP.* p. 172) : अत्राद्यमनुपपन्नत्वान्न ग्राह्यम् । – the former solution $x = \sqrt{20}$ being not feasible it is not accepted.

Remark : The solution $x = \sqrt{20}$ is not accepted by Kṛṣṇa presumably because it is not integral. Otherwise, it is also a right solution and one of the only two solutions possible.

We have the following formulae : Area of a triangle can be written as $= \frac{1}{2}\ ab\ sinC = \frac{1}{2}\ bc\ sinA = \frac{1}{2}\ ca\ sinB$, where A, B, C are the angles of the triangle and $a,\ b,\ c$ are the corresponding sides. Taking $\Delta = \frac{1}{2}\ ab\ sinC$ in the example given



$$4 = \frac{1}{2}\ .\sqrt{13}\ .\sqrt{5}\ .\ sinC$$

i.e. $\frac{8}{\sqrt{65}} = sinC$ or $sin\,(180 - C)$ (since $sinC$ is also equal to $sin\,(180 - C)$)

So we have two triangles ABC and $A'BC$ with $a,\ b,$ and angle C or angle $(180 - C)$ and the third side has two values c and c'.

Note : In this context Kṛṣṇa says that he would give the explanation for preferring the second solution, in the section on *madhyamāharaṇa* (*BG.* p. 172) : अनुपत्तौ उपपत्तिं तु मध्यमाहरणविवरणे वक्ष्यामः । However, no relevant material is found in that section to corroborate the statement.

Example : Another interesting example for which Kṛṣṇa offers his own solution is the following (*BG.* v. 124) :

घनैक्यं जायते वर्गो वर्गैक्यं च ययोर्घनः ।
तौ चेद्वेत्सि तदाऽहं त्वां मन्ये बीजविदां वरम् ॥

– Give two numbers such that the sum of their cubes is a square and the sum of their squares is a cube.

Kṛṣṇa says (*BP.* p. 180) : अथवान्यथा मया कल्पितौ राशी . . . – Otherwise assume as per my method – the two numbers to be $5x^3$, $10x^3$. अनयोः वर्गैक्यं स्वत एव जायते याघव १२५ – Squaring and adding, we have $25x^6 + 100x^6 = 125x^6 = (5x^2)^3$ a cube. Therefore the second requirement of the question is satisfied automatically.

Using the first requirement by cubing both and adding, $125x^9 + 1000x^9 = 1125x^9$.

If $1125x^9 = p^2$, (say for some p), then $15^2 . 5. x^8. x = p^2$, i.e. $(15x^4)^2 5x = p^2$ i.e. $5x$ should be a square giving $x = 5, 5^3, 5^5, . . .$ (any odd power of 5). So, the desired numbers are $5x^3$, $10x^3$ when x is any odd power of 5.

Note : The solution is thus not unique.[6]

In addition to Bhāskara's examples, Kṛṣṇa refers to two illustrations from the *Līlāvatī*[7] for the benefit of the students.

6. If we assume the numbers to be $2x^3$, $2x^3$, we have the first requirement *viz.*, the sum of the square $= 8x^6 = (2x^2)^3$, a cube, satisfied. The sum of the cubes is $16x^9$. Thus $16x^9$ needs to be a square which is the case when x is any square number. The numbers are $2x^3$, $2x^3$ for any number x which is a square making $x = 4$. Thus there are infinite number of such solutions when repetition is allowed.

7. (v.158) अस्ति स्तंभतले बिलं तदुपरि क्रीडाशिखंडी स्थितः स्तंभे हस्तनवोच्छ्रिते त्रिगुणितस्तंभप्रमाणान्तरे ।
दृष्ट्वाऽहिं बिलमाव्रजन्तमपतत् तिर्यक्सतस्योपरि क्षिप्रं ब्रूहि तयोर्बिलात्कतिमितैः साम्येन गत्योर्युतिः ॥

(v. 160) सखे पद्मतन्मज्जनस्थानमध्यंभुजः कोटिकर्णान्तरं पद्मदृश्यम् ।
नलः कोटिरेतन्मितं स्याद्यदंभो वदैवं समानीय पानीयमानम् ॥

11

5.4. *Saṅkramaṇa* (Concurrence)[8] – Definition :

It is interesting to note that Bhāskara has made use of *saṅkramaṇa* dealing with more than one variable in the context of *ekavarṇa samīkaraṇa* and *madhyamāharaṇa* relating to a single variable.

Saṅkramaṇa is basically the solution of the simultaneous equations,

$$x + y = a$$
$$x - y = b$$

According to Gaṅgādhara[9] (15th Cent. A.D.), the commentator on *Līlāvatī*, it is "the investigation of two quantities concurrent or grown together in the form of their sum and difference".

Brahmagupta's rule for the solution is (*Br.Sp.* XVIII. 36 ab) :

योगोऽन्तरयुतहीनो द्विहृत:संक्रमणमन्तरविभक्तं वा ।

– The sum is increased and diminished by the difference and divided by two ; the result will be the two unknown quantities. This is called concurrence.

In mathematical symbols, the above statement means

$$x = \frac{a+b}{2}, \quad y = \frac{a-b}{2}$$

Bhāskara in his *Līlāvatī* (p.63) gives the following definition for *saṅkramaṇa* which is precisely the definition given by Brahmagupta.

8. Datta and Singh, *op.cit.*, Vol. II, p. 43 :

This topic "commonly discussed by almost all Hindu writers goes by the special name *saṅkramaṇa* (concurrence). According to Nārāyaṇa (1350) it is also called *saṅkrama* or *saṅkrāma*. Brahmagupta (628) includes it in algebra while others consider it as falling within the scope of arithmetic".

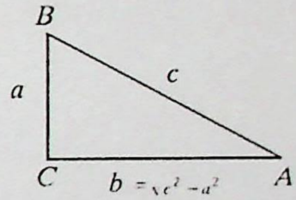
9. *loc.cit.*

योगो अन्तरेण ऊन युत अर्धितः तौ राशी स्मृतौ संक्रमणाख्यमेतत् ।

– Suppose we have to find two numbers whose sum and difference are given. Then add and subtract the given numbers and divide them by 2 to get the two numbers.

5.4.1. *Saṅkramaṇa* - Examples :

The following two examples illustrate Bhāskara - Kṛṣṇa's application of *saṅkramaṇa.*



Example 1 : (*BG.* v. 148) :

भुजात्र्यूनात्पदं व्येकं कोटिकर्णान्तरं सखे ।
यत्र तत्र वद क्षेत्रे दो:कोटिश्रवणान्मम ।।

– Find the three sides of right angled triangle given that *karṇa* (hypotenuse) less *koṭi* (vertical side) = $\sqrt{bhuja\,(\text{base}) - 3} - 1$.

Kṛṣṇa explains the method in the following manner (*BP.* p. 199) : Let c be the hypotenuse (*karṇa*) , b the base (*bhuja*) and a the vertical side (*koṭi*) of the $\Delta\, ABC$. It is given that $\sqrt{b-3} - 1 = c - a$

then $\qquad \sqrt{b-3} = c - a + 1 \qquad$ ①

Kṛṣṇa assumes $(c - a)$ to be 2

then equation ① becomes $\sqrt{b-3} = c - a + 1 = 2 + 1 = 3$

Squaring both sides

$$b - 3 = 9$$

i.e. $$b = 9 + 3 = 12.$$

Since $b^2 = c^2 - a^2$ we have $c^2 - a^2 = 12^2 = 144$.

$$\frac{c^2 - a^2}{c - a} = c + a \quad \text{i.e.} \quad \frac{144}{2} = c + a = 72$$

Kṛṣṇa explains the use of *saṅkramaṇa* as : "कल्पितकोटिकर्णान्तरमिदं २ . . . कोटिकर्णयोग ७२. योगान्तराभ्यां योगान्तरेणोनयुतोर्द्धित इति संक्रमणसूत्रेण जातौ कोटिकर्णौ ३५, ३७ ।"

i.e.

$$\text{कर्ण} - \text{कोटि} = 2$$

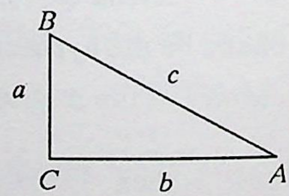
$$\text{कर्ण} + \text{कोटि} = 72$$

$$\text{कर्ण} = \frac{72+2}{2} = 37$$

$$\text{कोटि} = \frac{72-2}{2} = 35$$

Kṛṣṇa concludes that any chosen value of $c - a$ will give a corresponding value for $c + a$ from which a, b and c. can be found out. एवमनेकधा । – Thus there can be many solutions.

<u>Example 2 :</u> (*BG.* v. 150) :



चत्वारिंशद्युतिर्येषां दोः कोटिश्रवसां वद ।
भुजकोटिवधो येषु शतं विंशतिसंयुतम् ।।

– The sum of the three sides of a right angled triangle is 40 and the product of two sides which contains the right angle is 120. Find the three sides.

According to the method given by Kṛṣṇa (*BP.* pp. 203-04), let c be the hypotenuse (*karṇa*) , b the base (*bhuja*) and a the vertical side (*koṭi*) of the $\Delta\, ABC$. It is given that $ab = 120$.

It is known that $(a + b + c)\,(a + b - c) = (a + b)^2 - c^2 = a^2 + b^2 + 2ab - c^2$
$= 2ab$ (since $a^2 + b^2 = c^2$)

So $(a+b+c)(a+b-c) = 2 \times 120 = 240$

i.e. $a+b-c = \dfrac{240}{a+b+c} = \dfrac{240}{40} = 6$

Thus

भुजकोटिकर्णयोग: $= 40$

भुजकोटियुतिकर्णान्तरं $= 6$

Kṛṣṇa assumes *bhuja* + *koṭi* as one *rāśi* and *karṇa* as the other (*BP.* p. 204) : भुजकोटियुतिरेको राशि: कर्णोऽपर: ।

Using *saṅkramaṇa* we have

भुज + कोटि $= \dfrac{40+6}{2} = 23$

कर्ण $= \dfrac{40-6}{2} = 17$

From the well-known identity $(a-b)^2 = (a+b)^2 - 4ab$,

$$(a-b)^2 = 23^2 - 4 \times 120 = 529 - 480 = 49$$

i.e. कोटि – भुज $= 7$

Again

कोटि + भुज $= 23$

Using *saṅkramaṇa* कोटि $= \dfrac{23+7}{2} = 15$

भुज $= \dfrac{23-7}{2} = 8$

Kṛṣṇa concludes (*BP.* p. 204) : भुजकोटियोरन्तरं ७ । भुजकोटियोगश्चायं २३ । आभ्यां संक्रमणेन जाते भुजकोटि ८, १५. – The difference of *bhuja* and *koṭi* is 7 ; their sum is 23 ; therefore by the method of concurrence *bhuja* is 8 and *koṭi* is 15.

5.5. *Madhyamāharaṇa* - Solution of Quadratic Equation :

From a very early age, Indians were aware of quadratic equations.

5.5.1. Definition :

As Datta and Singh (p. 59) observe : "the altar – construction of the Vedic Hindus involved the solution of the complete quadratic equation $ax^2 + bx = c$, as well as the pure quadratic $ax^2 = c$. The equation that had to be solved was $7x^2 + \frac{1}{2}x = 7\frac{1}{2} + m$ which gives $x = \frac{1}{28}\left(\sqrt{841 + 112m} - 1\right)$. . . ".

Madhyamāharaṇa is the technical name given by early Indian algebraists to solve quadratic equations. Since these equations also deal with higher powers of one variable only, Kṛṣṇa rightly says it belongs to *ekavarṇa samīkaraṇa*.

Bhāskara himself defines *madhyamāharaṇa* as (*BG.* p. 59) : तच्च मध्यमाहरणमिति व्यावर्णयन्त्याचार्याः । यतोऽत्र वर्गराशावेकस्य मध्यमस्याहरणमिति । – It is specifically named by the Ācāryas as the *madhyamāharaṇa,* because, the middle term of the quadratic gets removed.

The origin of the name can be found in the principle underlying the method. Generally the quadratic equation has three terms. The general method of completing the square, reduces it to two terms only. Thus the middle term (*madyama*) *viz.*, the first degree term, is eliminated (*āharaṇa*) for solving the equation. Hence the name.

Quadratic equations and their solutions were known to mathematicians before Bhāskara. The *BM* (p. 191, III.9(5′)) has a few problems of this type. Similarly Āryabhaṭa I (*Āryabaṭīya,* II.20) and Brahmagupta (*Br.Sp.* XVIII, vv. 44-5) deal with quadratic equations. But the most significant contribution came from Śrīdharācārya whose treatise unfortunately is lost to us. He gives an excellent method for solving the

quadratic equation which is quoted by Bhāskara. In almost all the examples given by Bhāskara, Śrīdhara's formula is used. Even to this day, school children everywhere learn to find the roots of the quadratic equation using Śrīdhara's rule, without being told so.

5.5.2. Śrīdhara's Rule for solution of a Quadratic Equation (*BP.* p. 188) :

चतुराहतवर्गसमरूपैः पक्षद्वयं गुणयेत् ।
पूर्वव्यक्तस्य कृतेः समरूपाणि क्षिपेत् तयोरेव ॥[10]

– multiply both sides by four times the co-efficient of the square of the unknown and add to both sides the square of the co-efficient which belongs to the unknown before the multiplication, then extract the square root.

The special feature of the treatment of quadratic equation by the Indian algebraist is that they kept c on the r.h.s. i.e. taking the equation as $ax^2 + bx = c$, so that the discriminant $4ac + b^2$ of the quadratic always remains positive.

Kṛṣṇa paraphrases Śrīdhara's Rule thus :

To find the roots of the equation $ax^2 + bx = c$, multiplying by $4a$ throughout,

$$4a^2x^2 + 4abx = 4ac$$

10. This reading of the verse as given by Kṛṣṇa has been found in the text of Rāmakṛṣṇa, another commentator of *BG.* Colebrooke and Datta and Singh accept this reading. This is the reading preferred in this study. Here the word *kṣipet* denotes addition. ;

Cf. Monier Williams, *A Sanskrit - English Dictionary*, p. 328. : the root *'kṣip'* means 'to add' (math.).

Sūryadāsa in his *Sūryaprakāśa* (I.O, 1533a, fol. 44a-b) gives a different reading for the second line as : अव्यक्तवर्गरूपैः युक्तौ पक्षौ ततो मूलम् । According to Datta and Singh (Vol. II, p. 65) Jñānarāja (15-16 Cent. A.D.) also has given the same reading in his *Bījagaṇita*. This is accepted by Mm. Sudhakara Dvivedi as is found in his edition of *BG* (p. 61).

Adding b^2 to both sides in order to complete the square,

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2$$

$$(2ax + b)^2 = 4ac + b^2$$

$$2ax + b = \sqrt{4ac + b^2}$$

$$2ax = \sqrt{4ac + b^2} - b$$

$$x = \frac{\sqrt{4ac + b^2} - b}{2a}$$

Note : It is tacitly assumed that a, b, c are all positive.

5.5.3. Special Features of the Quadratic Equation :

Below is the discussion, as given by Bhāskara and elucidated elaborately by Kṛṣṇa on the special features of the quadratic equation. Illustrations are taken up to 1) show that every quadratic equation has two roots, 2) find when the two roots are positive, 3) see when there are two positive roots, one may not be desirable for a particular problem and 4) see when the negative root cannot be accepted.

5.5.3.1. Two roots of the Equation :

The Indian algebraists were aware that the quadratic equations generally have two roots. Both Mahāvīra in his *GSS* (IV. 57,59) and Brahmagupta in his *Br.Sp.* (XVIII, 49-50) give examples having two roots. Bhāskara observes that one or two solutions is accepted, when he says : (*BG.* v. 132) :

अव्यक्तमूलर्णगरूपतोऽल्पं व्यक्तस्य पक्षस्य पदं यदि स्यात् ।
ऋणं धनं तच्च विधाय साध्यमव्यक्तमानं द्विविधं क्वचित्तु ।।

– If the square root of the known side (of the quadratic) be less than the negative absolute term occurring in the square root of the unknown side, then making it negative as well as positive, sometimes, two values of the unknown may be possible.

Note : In the modern treatment, the fact that $4ac + b^2$ has two square roots, one positive and one negative is used by attaching the symbol '±' to $\sqrt{4ac + b^2}$. In Bhāskara's rule it is noted that $(2ax + b)^2 = \{-(2ax + b)\}^2$. Clearly, because of the concern for only practical problems to be solved, the cases where $\sqrt{4ac + b^2} - b$ and $-\sqrt{4ac + b^2} - b$ are both positive, are considered. A necessary condition therefore would be $b < 0$ when $a, c > 0$. Bhāskara's rule says that b should be numerically greater than $\sqrt{4ac + b^2}$ for such a situation.

<u>Example</u> (*BG.* v. 141) :

वनान्तराले प्लवगाष्टभागः संवर्गितो वल्गति जातरागः ।
बूत्कारनादप्रतिनादहृष्टा दृष्टा गिरौ द्वादश ते कियन्तः ॥

– One eighth of a troop of monkeys, squared, was sporting in the forest ; the remaining twelve could be seen on the hill, chattering to one another. How many monkeys were there in all?

Kṛṣṇa discusses this example to elucidate : 1) every quadratic equation has two roots ('*dvividham*') and 2) that the two roots may be positive.

If x is the total number of monkeys,

the equation is $$\left(\frac{x}{8}\right)^2 + 12 = x \quad \text{or} \quad \frac{1}{64}x^2 + 12 = x,$$

i.e. $$x^2 - 64x = -768$$

and we have by the procedure in Kṛṣṇa's treatment of Śrīdhara's rule **(5.5.1.)**.

Adding to both sides, $\left(\frac{64}{2}\right)^2 = 1024$

$$x^2 - 64x + 1024 = 1024 - 768$$

$$(x-32)^2 = 256$$

$$(x-32)^2 = 16^2$$

Bhāskara's rule (*BG.* v. 132) अव्यक्तमूलर्ण...। , which precedes the example gives the solution as

$$\pm\ (x-32) = 16$$

After equating both sides, the equation reduces to

$$x - 32 = 16$$

or $$-x + 32 = 16$$

giving $$x = 48,\ 16$$

Both the above solutions are acceptable.

Since the constant term on the side of the unknown (l.h.s.) is numerically greater than the constant term on the other side (r.h.s.), both the roots are positive in this case.

5.5.3.2. Positive Roots :

Kṛṣṇa anticipates the question that (*BP.* p. 186): ननु अव्यक्तपदरूपेभ्यो व्यक्तपदेऽधिकेऽपि द्विविधं मानम् अनया युक्त्या कथं न स्यात् । – If it should be asked "if the square root of the known (side) is greater than the absolute number in the square root of the unknown (side), too, why will there not be two solutions?" and explains – अव्यक्तपक्षजरूपाणाम् ऋणत्वेऽव्यक्तस्य धनत्वमेव । अस्मिन् प्रकारेऽव्यक्तशेषस्य धनत्वार्थमव्यक्तपक्षरूपाण्येव व्यक्तपक्षात् शोध्यानि । तानि च धनं भवतीति

नास्ति अनुपपत्तिः । – If the constant produced from the unknown side (l.h.s.) is negative, this implies that the unknown is positive. Thus, for the positiveness of the remainder of the unknown, the constant of the unknown side (l.h.s.) is deducted from the known side (r.h.s.). This amounts to adding the absolute numerical value of the negative constant. There is no inconsistency since both become positive in the r.h.s.

5.5.3.3. Positive Roots - Practicability :

Even in case there are two positive roots, one of the roots may not suit the practical problem posed[11], as discussed now.

Example (*BG.* v. 143) :

कर्णस्य त्रिलवेनोना द्वादशाङ्गुलशंकुभा ।
चतुर्दशाङ्गुला जाता गणक ब्रूहि तां द्रुतम् ॥

– The shadow of a gnomon of twelve *aṅgulas* in length being lessened by the third part of the hypotenuse, became fourteen *aṅgulas* in length. Tell me quickly, mathematician, the length of the shadow.

Here, after forming the equation (using Pythagoras theorem) and solving it, the two solutions for the length of the shadow are $\frac{45}{2}$ and 9. Since the length of the shadow should be more than 14, according to the details given, the solution 9 cannot apply.

5.5.3.4. Negative Roots :

Consider the equation $(x - 16)^2 = (32)^2$. Kṛṣṇa adds (*BP.* p. 186) : अथ रूपाणां धनत्वेऽव्यक्तस्यर्णत्वमेवेति द्वितीयप्रकारे अव्यक्तमेव धनत्वार्थमितरपक्षात् शोध्यम् । व्यक्तरूपाणि तु अव्यक्तपक्षजपदरूपेभ्यः शोध्यत्वादृणं भवति । तानि यद्यधिकानि तदा ऋणं मानं स्यादिति द्वितीयं सर्वाप्यनुपपन्नम् । – But in the second case, where, the number being positive, the unknown is negative, the unknown must be subtracted

11. Refer Note under **5.3.2** ; most probably this is the section where Kṛṣṇa should have given a reason for his earlier remark.

from the other side of the equation, to become affirmative; and the number on the absolute side, being subtracted from that comprised in the root extracted from the unknown side, becomes negative; and, if it be the greater of the two, the value is negative. The second value consequently is every way incongruous.

Kṛṣṇa concludes : अत उक्तं अव्यक्तमूलर्णगरूपतोल्पं व्यक्तस्य पक्षस्य पदं स्यात् । – thus the essential condition for the equation to have two positive roots is for the constant term on the side of the unknown (l.h.s.) to be greater than the constant term on the side of the known (r.h.s.).

This observation relates to the stage later to the application of Śrīdhara's procedure, *viz.*, completing the square by adding a suitable constant.

5.5.3.5. Negative roots - impracticability :

Both Bhāskara and Kṛṣṇa rule out negative roots though they can be allowed in numerical examples. However it should be remembered that Indian mathematicians used algebra to solve practical problems, by accepting practical positive solutions and ignoring impractical negative solutions. Hence Bhāskara (*BG.* p. 66) says : न हि व्यक्ते ऋणगते लोकस्य प्रतीतिरस्तीति ।

The example taken up to illustrate the above is (*BG.* v. 142) :

यूथात्पञ्चांशकस्त्र्यूनो वर्गितो गह्वरं गतः ।
दृष्टः शाखामृगः शाखामारूढो वद ते कति ।।

– The fifth part of a troop of monkeys less three, and squared, went inside a cave ; one monkey however was in sight having climbed on the branch (of a tree). How many monkeys were there?

If x is the total number of monkeys, after forming the quadratic equation and applying *madhyamāharaṇa*, we have the solutions for x as 50, 5.

Kṛṣṇa argues that only the value $x = 50$ can be taken as the answer and not $x = 5$, following the earlier comment of Bhāskara on negative solutions. The question clearly says that "the fifth part of a troop of monkeys leaving out 3 . . ." $\left(\frac{1}{5} \times 50\right) - 3$ is possible but $\left(\frac{1}{5} \times 5\right) - 3$ gives negative value, hence not allowed ; राशिश्च ५ नह्यस्य पञ्चांश १ स्त्रिभिरूनः संभवति । He further adds – एवमस्मिन्नेवोदाहरणे यूथात्पंचांशकस्त्रिच्युत इति यद्यालापः स्यात्तदा द्वितीयमानमेव युक्तं न तु पूर्वम् । – If the example had been "the fifth part of the troop of monkeys subtracted from three", then the second solution would be correct one and not the first.

Remarks : Mathematically both the roots are admissible.[12]

5.5.3.6. Exceptions :

Bhāskara has used the word *kvacit* in his verse (*BG.* v. 132) to emphasize that in all cases two roots may not be acceptable. Kṛṣṇa corroborates Bhāskara's statements and establishes the logic behind Ācārya's views and the specific use of the word *kvacit*. (*BP.* p. 187) : क्वचित् क्षेपशोधनादिना शेषविधिना विपरीतमपि भवति । क्वचित् अव्यक्तस्य स्वतोपि ऋणत्वे द्विविधमूलसंभवेऽपि द्वितीयमनुपपन्नं भवति । अत एव आचार्यैर्द्विविधं क्वचित् इति अनियमेनैवोक्तम् । – "It is thus

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12. Corroborating the views of Bhāskara and Kṛṣṇa, Mm. Sudhakara Dvivedi observes (*BG.* p.60) :
स्वमूले धनर्णे भवत इति युक्त्या व्यक्तपक्षमूलं धनं वा ऋणं विधाय यद्यव्यक्तमानं साध्यते तदा सर्वदा तन्मानं द्विविधं भवति परन्तु ऋणस्य लोके ह्यप्रसिद्धत्वाद्यदा मानद्वयं धनमेवागच्छति तदेवाव्यक्तमूलर्णगरूपतोऽल्पे व्यक्तपक्षपदे द्विविधं मानमाचार्येणानीतमिति – Every positive square has two roots, negative and positive. Therefore every quadratic equation has two roots. But since a negative root is *aprasiddha*, Ācārya does not accept a negative root.
According to Muralidhar Jha, however, the words, "द्वितीयमत्र न ग्राह्यमनुपपन्नत्वात् । न हि व्यक्ते ऋणगते लोकस्य प्रतीतिरस्तीति" are themselves extrapolation. For, though $\left(\frac{1}{5}x - 3\right)$ is negative where x is taken to be five, its square i.e. $\left(\frac{1}{5}x - 3\right)^2 = 4$ is positive. And therefore 5 can also be a solution according to him. ". . . एतत्कोष्टान्तर्गतं पदं प्रक्षिप्तमेव यतो यदि यूथप्रमाणम् ५, कल्प्यते तदा पंचांशः १, त्र्यूनः २ं, वर्गितः ४, अत्र नानुपपन्नत्वम् ।"

in many various ways. Sometimes by subtraction of the addition or other means it is the reverse. Sometimes, by reason of the unknown being naturally negative, though the root might possibly be two-fold, the second is incongruous. Accordingly the author has said indefinitely 'This holds in some cases'''.[13]

To elucidate this, Kṛṣṇa draws upon the *Praśnādhyāya* of *Siddhānta Śiromaṇi* of Bhāskara and states : (*BP.* p. 188) : "· · · द्युज्यका . . . चेत् श्रुतम्'' इत्यस्मिन् प्रश्नोदाहरणे . . . समशोधने कृते पक्षयो: पदग्रहणावसरे अव्यक्तमृणं रूपाणि धनं इत्येव गृह्यते । अत एव तदानयनसूत्रेपि तेन द्वियूनो भवेदित्येवोक्तम् ।. . . एवं मदुक्तयुक्त्या द्विविधमानोपपत्त्यनुपपत्ती सर्वत्रावधार्ये । तदेवमुपपन्नं द्विविधं क्वचिदिति ।। – "See an instance in the chapter on the three problems where the question is proposed as one requiring the resolution of a quadratic equation; and the answer shows, that, in taking the roots of the two sides of the equation, the unknown has been taken negative and the absolute number positive ; for if the number were taken negative, the answer would come out differently. Thus by the reasoning here set forth, the congruity or incongruity of a two-fold value is to be everywhere understood ; and the author's remark, 'it holds in some cases', is justified''.[14]

Thus Kṛṣṇa lays emphasis on the use of the word '*kvacit*' in the *sūtra* given by Bhāskara.

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13. Tr. by H.T. Colebrooke, *op. cit.*, p.209.
14. *ibid.*, pp. 209-10.

Jivanatha Jha (p.342) is also of the same opinion : आचार्येण तु अव्यक्तमूलर्णगरूपतोल्पमिति पद्ये द्विविधं क्वचित्स्यादिति विशेषणमतो दत्तं , तथा च सिद्धान्ते द्युज्यापक्रमभानुदोर्गुणयुतिपद्योत्पत्तौ समशोधनादेरुत्तरं पक्षयोर्मूले गृहीते अव्यक्तपक्षमूलरूपमृणमायाति । तत्र धनमिति गृहीतं यतस्तेनाद्य ऊनो भवेदित्युक्तं रूपाणामृणत्वे तु तेनाद्य आढ्यो भवेदित्युक्तिः स्यात् । अतस्तत्र द्वितीयमानमेव युक्तं प्रथममानं अनुपपन्नमेवं यथासंभवं क्वचिदिति विशेषणं योजनीयम् । - i.e. Bhāskara has specifically used the phrase 'on certain occasions'. For this, the example '*dyujya . . . cet śrutam*' can be considered from *Siddhānta Śiromaṇi* , where though there are two solutions, only the second solution has been taken into account. '

5.6. Other Examples :

In the same chapter, Bhāskara proves certain known results using geometrical figures. Kṛṣṇa takes up an example of Bhāskara and discusses it with diagrammatic representation:

Example 1 : Diagrammatic representation (*BG.* v. 147) :

दो: कोटयन्तरवर्गेण द्विघ्नो घात: समन्वित: ।
वर्गयोगसम: स स्याद् द्वयोरव्यक्तयोर्यथा ।।

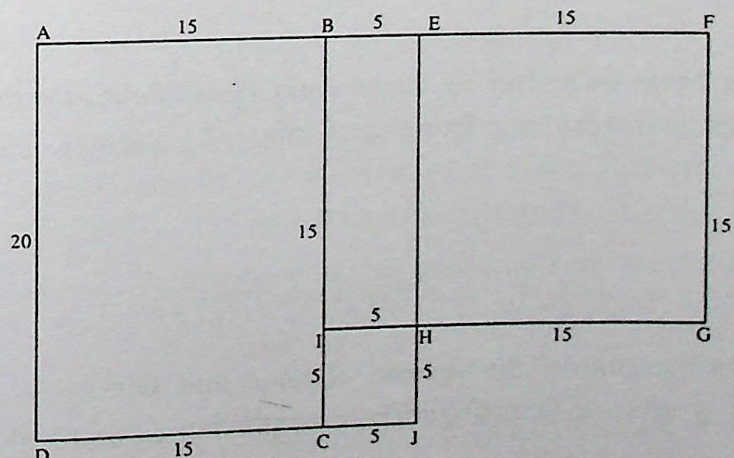
– Twice the product of the altitude and base being added to the square of their difference, is equal to the sum of their squares, just as with any two unknown quantities (instead of altitude and base),

Let x be the altitude ($x =$ कोटी) and y be the base ($y =$ दो:) then

$$(x - y)^2 + 2xy = x^2 + y^2$$

which is a well-known identity. For this identity Kṛṣṇa gives his own figure (*BP.* p. 199) choosing $x = 20$, $y = 15$.

Draw Rectangle *ABCD* 15 × 20 square units. Extend *DC* and *AB* by 5 units. Join *EJ* to complete the square *AEJD* which would be 20 × 20 square units. Take points *I* and *H* on *BC* and *EJ* so that *CI* and *JH* are



5 units each. Now $BI = EH = 15$ units. Complete the square $EFGH$ which is equal to 15×15 square units.

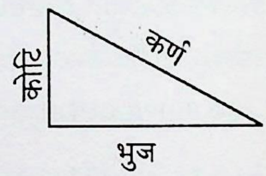
Now, sq. $AEJD$ + sq. $EFGH$ = sq. $IHJC$ + Rect. $ABCD$ + Rect. $BFGI$.

i.e. $20^2 + 15^2 = 5^2 + (15 \times 20) + (15 \times 20)$

Illustrating with $x = 20$, $y = 15$, the identity is established :

i.e. $x^2 + y^2 = (x - y)^2 + 2xy$ or $(x - y)^2 + 2xy = x^2 + y^2$

<u>Example 2 :</u> (*BG.* v. 150) :



चत्वारिंशद्युतिर्येषां दोःकोटिश्रवसां वद ।
भुजकोटिवधो येषु शतं विंशतिसंयुतम् ।।

– To find the three sides of the triangle, given that the product of *bhuja* (*doḥ*) and *koṭi* is 120 and the sum of the three sides *bhuja, koṭi* and *karṇa* is 40.

Note : The above example was dealt with in **5.4.** while explaining *saṅkramaṇa.* Here its taken up to establish certain well known identities.

Bhāskara solves it in the following manner which Kṛṣṇa explains (*BP.* p. 202) :

Let *bhuja,* be x, *koṭi* be y and *karṇa* be z. Following the *sūtra,* वर्गयोगस्य यद्राश्योर्युतिवर्गस्य चान्तरं द्विघ्नघातसमानं स्यात् ।, i.e. using the identity,

$$(x + y)^2 - (x^2 + y^2) = 2xy$$

i.e. $$(x + y)^2 - z^2 = 2xy = 2 \times 120 = 240$$

since z^2 is the sum of the squares of *bhuja* and *koṭi* $(x^2 + y^2)$ says Kṛṣṇa (*BP.* p. 204) : यो हि भुजकोटिवर्गयोगः स एव पूर्वकर्णवर्गः । Considering $(x + y)$

as a single number and z as another number Kṛṣṇa says : अनयोर्वर्गान्तरं तच्च योगान्तरघातसमं – The difference of the squares of the above two numbers is the product of the sum and difference of the two numbers (*viz.*, the identity $a^2 - b^2 = (a + b)(a - b)$).

So, $$(x + y)^2 - z^2 = 240$$

i.e. $$(x + y - z)(x + y + z) = 240$$

But it is given in the example that $x + y + z = 40$. So,

$$(x + y - z)(40) = 240$$

$$\text{or } x + y - z = \frac{240}{40} = 6.$$

There are now two equations

$$x + y + z = 40 \quad ①$$

$$x + y - z = 6 \quad ②$$

Then Kṛṣṇa says (*BP.* p. 204) : अथ योगान्तराभ्यामेताभ्यां ४०/६ संक्रमणेन जातौ राशी २३/१७ भुजकोटियुतिरेकः २३ कर्णोऽपरः १७ अत्र लघुराशिरेव कर्णो ज्ञेयः । भुजकोटियुतितस्तस्याधिक्यासंभवात् । उक्तमप्याचार्यैः लीलावत्याम्[15] –

धृष्टोद्दिष्टमृजुभुजं क्षेत्रं यत्रैकबाहुतः स्वल्पा ।
तदितरभुजयुतिरथवा तुल्या ज्ञेयं तदक्षेत्रम् ।।

Kṛṣṇa says that by equating simultaneously (see **5.4.1.** *Saṅkramaṇa*) we get two values for $x + y$ and z (23, 17) respectively. The smaller one should be taken as *karṇa* (z) because the sum of *bhuja* and *koṭi* should be greater than *karṇa*. It is said in the *Līlāvatī* that it is impossible for one side of a triangle or a polygon to be greater than the sum of the other sides.

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15. *Līlāvatī*, v. 169

$$y + x = 23$$

$$z = 17$$

$$(x - y)^2 = (x + y)^2 - 4xy = (23)^2 - (4 \times 120) = 529 - 480 = 49$$

So, $x - y = 7$

by *saṅkramaṇa* again $x = 15$

$$y = 8$$

Thus the three sides of a triangle are *koṭi* = 15, *bhuja* = 8 and *karṇa* = 17.

In the main, it is shown that Kṛṣṇa has cleverly reduced some of the equations of two or more variables to that of one. The world acclaimed Śrīdhara's rule on quadratic equation is illustrated elaborately by Kṛṣṇa. The various possibilities regarding roots of the quadratic equation have also been discussed.

CHAPTER - 6
ANEKAVARṆA SAMĪKARAṆA – MADHYAMĀHARAṆA – BHĀVITA

Anekavarṇa samīkaraṇa means equations with more than one unknown. There are both simple and multiple equations of this type. Some variations of these are discussed by Bhāskara and explained in detail by Kṛṣṇa. We find the earliest treatment of equations of several unknowns by the Indian algebraists, in the *BM.* Later Mahāvīra in his *GSS* and Brahmagupta in his *Br.Sp.* also deal with such equations.

6.1. *Anekavarṇa samīkaraṇa* or Linear Equation with many unknowns :

The *sūtras* explaining the method employed to solve the equations involving several unknowns are commented on by Bhāskara himself in his excellent commentary. Kṛṣṇa says (*BP.* p. 205) : एतानि सूत्राणि आचार्यैरेव सम्यक् व्याख्यातानीति नास्माभिर्व्याक्रियन्ते ।

Example 1 : To substantiate Bhāskara's statement that *kuṭṭaka* may be employed to get integral solutions, Kṛṣṇa takes the following example (*BG.* v. 159) :

षड्भक्तः पञ्चाग्रः पञ्चविभक्तो भवेच्चतुष्काग्रः ।
चतुरुद्धृतस्त्रिकाग्रो द्व्यग्रस्त्रिसमुद्धृतः कः स्यात् ॥

– Find the number which when divided by 6 leaves the remainder 5, when divided by 5 leaves 4, when divided by 4 leaves 3, and when divided by 3 leaves 2 as remainder.[1]

Let N be the required number. According to the statement of the problem

$$N = 6x + 5 = 5y + 4 = 4z + 3 = 3w + 2$$

1. Note that the hypothesis that the number divided by 6 leaves remainder 5 implies that it leaves remainder 2 when divided by 3, renders the latter hypothesis, a redundant one.

Taking the last of the three equations given by the above relation

$$4z+3 = 3w+2$$
$$4z = 3w-1$$

Kṛṣṇa says : इयमन्त्या । अन्त्योन्मितौ कुट्टकविधेर्गुणाप्ती । i.e. the last two equations may be solved for integral solutions using *kuṭṭaka.*

$$4z+1 = 3w$$

Using *kuṭṭaka* we have solution sets, (2,3) (5,7) (8,11) . . . One general solution is

$$\therefore \quad w = 4t+3$$
$$z = 3t+2 \text{ where } t = 0, \pm 1, \pm 2 \ldots$$

Taking the second equation

$$5y+4 = 4z+3$$

i.e. $$5y = 4z-1$$
$$= 4(3t+2)-1$$
$$= 12t+7$$

i.e. $$5y = 12t+7$$

Kṛṣṇa remarks that (*BP.* p. 208) : अतोऽत्र भूयः कुट्टकः कार्यः using *kuṭṭaka* again, we have solution sets (11,4) (23,9) . . . So, one set of solutions is

$$y = 12s+11$$
$$t = 5s+4 \text{ where } s = 0, \pm 1, \pm 2 \ldots$$

Now from the first of the three equations

$$x = \frac{5y-1}{6} = \frac{5(12s+11)-1}{6}$$
$$= \frac{60s+54}{6} = 10s+9$$
$$y = 12s+11$$
$$z = 3t+2 = 3(5s+4)+2 = 15s+14$$
$$w = 4t+3 = 4(5s+4)+3 = 20s+19$$

Thus

$$x = 10s + 9$$
$$y = 12s + 11$$
$$z = 15s + 14$$
$$w = 20s + 19$$

Now, it is given that $N = 6x + 5 = 5y + 4 = 4z + 3 = 3w + 2$. Thus to find out N we can substitute for any one of the values, e.g., $x = 10s + 9$. Then

$$N = 6x + 5 = 6(10s + 9) + 5 = 60s + 59$$
$$y = 12s + 11$$
$$N = 5y + 4 = 5(12s + 11) + 4 = 60s + 59$$

To verify using $z = 15s + 14$

$$N = 4z + 3 = 4(15s + 14) + 3 = 60s + 59$$

[2]Giving different values for $s = 0, \pm 1, \pm 2 \ldots$, we get different values for N. When $s = 0$, $N = 59$; $s = 1$; $N = 119 \ldots$

<u>Example 2 :</u> (*BP.* p. 210) : एकस्मिन् पक्षे एकमेव अव्यक्तं स्यात् अन्यत्र च रूपाण्येव स्युः । तदा तस्य अव्यक्तस्य मानं सुबोधम् ।। – Keeping one variable on one side and the absolute numbers (*rūpāṇi*)on the other side makes the evaluation of the unknown easily comprehensible.

Example : (*BG.* v. 157) :

अश्वाः पञ्चगुणाङ्गमङ्गलमिता येषां चतुर्णां धना-
न्युष्ट्राश्च द्विमुनिश्रुतिक्षितिमिता अष्टद्विभूपावकाः ।
तेषामश्वतरा वृषा मुनिमहीनेत्रेन्दुसंख्याःक्रमात्
सर्वे तुल्यधनाश्च ते वद सपद्यश्वादिमौल्यानि मे ।।

– The first trader had 5 horses, 2 camels, 8 mules and 7 oxen ; the second had 3 horses, 7 camels, 2 mules and 1 ox ; the third had

2. An easier method, is to find the L.C.M. of 6, 5, 4, 3 which is 60. Therefore, the required N is equal to $60 - 1 = 59$.

6 horses, 4 camels, 1 mule and 2 oxen and the fourth had 8 horses, 1 camel, 3 mules and 1 ox. All of them were equally wealthy after selling these at the same rate. Find the price of a horse, a camel, a mule and an ox.

Note : The numbers have been expressed through a mixture of explicit mention and *bhūtasaṅkhyā*.

Let the price of a horse be 'a', a camel be 'b', a mule be 'c' and an ox be 'd'. Then the wealth of the first, second, third and fourth person are respectively

$$W_1 = 5a + 2b + 8c + 7d \quad ①$$

$$W_2 = 3a + 7b + 2c + d \quad ②$$

$$W_3 = 6a + 4b + c + 2d \quad ③$$

and

$$W_4 = 8a + b + 3c + d \quad ④$$

In other words, we have the set of equations

$$5a + 2b + 8c + 7d = 3a + 7b + 2c + d$$
$$= 6a + 4b + c + 2d = 8a + b + 3c + d \quad ⑤$$

which is a system of three linear equations in four unknowns. The equations are also homogeneous.

Kṛṣṇa says (*BP.* p. 210): अत्रैकपक्षे यथैकमेवाव्यक्तं भवति तथा यतितव्यम् । तत्रैकतरपक्षे एकं वर्णं विहाय यदवशिष्यते तत्तुल्यं चेदुभयोः पक्षयोः शोध्यते तर्ह्येकस्मिन्पक्षे एकमेवाव्यक्तं स्यात् । यं विहायाऽवशिष्टं शोध्यते तस्मिन्पक्षे तस्यैव वर्णस्य शेषत्वात् । तत्र कं वर्णं अपहाय शेषं पक्षयोः शोध्यमिति यद्यपि नास्ति नियमः तथापि प्रथमातिक्रमे कारणाभावात्प्रथमवर्णमपहाय शेषं पक्षयोः शोध्यम् । – following the *sūtras* (*BG.* vv. 152-54), one unknown alone is kept on one side and rest on the other side. There is no fixed rule as to which unknown should be retained on one side and which others should be taken to the other side. However, since there is

no reason to omit the first term, we take the first unknown on one side. From $W_1 = W_2$ we have

$$2a = 5b - 6c - 6d$$

$$\therefore \quad a = \frac{5b - 6c - 6d}{2} \qquad ⑥$$

From $W_2 = W_3$, we obtain

$$3a + 7b + 2c + d = 6a + 4b + c + 2d$$

$$3a = 3b + c - d$$

$$a = \frac{3b + c - d}{3} \qquad ⑦$$

$W_3 = W_4$, yields

$$2a = 3b + 2c - d$$

$$a = \frac{3b + 2c - d}{2} \qquad ⑧$$

⑥, ⑦ and ⑧ above give a system of equations equivalent to ⑤

BP. p. 211 : एवं प्रथमतृतीययोः प्रथमचतुर्थयोर्द्वितीयचतुर्थयोश्च समशोधनेन अन्या अपि यावत्तावदुन्मितयः संभवन्ति । परं प्रयोजनाभावान्न कृताः – Similarly we can equate, W_1 and W_3, W_1 and W_4, W_2 and W_4 and get other evaluations (*unmiti*) of *yavat tavat* (here *a*); but actually we do not get them because there is no use of them.

Kṛṣṇa discusses in detail, here, how the way suggested by Bhāskara is the desirable one as against other methods that are possible. For instance, one may fix up integer values for two of the variables and get a linear equation for the other two variables, use the *kuṭṭaka* process to get integer solutions for the two unknown variables. In this case we do get a set of integer solutions for *a*, *b*, *c*, *d*. But this process, does not exhaust the solutions. Kṛṣṇa adds that (*BP.* p. 211) : अथ भाज्यवर्णानां कालकादीनामिष्टानि मानानि प्रकल्प्य ऐक्यं कृत्वा स्वहरेण यदि ह्रियते तदा भिन्नमभिन्नं वा प्रथमवर्णमानं स्यात् । इतरेषां तु कल्पितान्येव । . . .अथ यद्यभिन्नमेव मानमपेक्षितं तर्हि यं कंचिदेकं वर्णं विहाय परेषां मानानीष्टानि कल्प्यानि । तथा सति भाज्ये एको वर्णः कानिचित् रूपाणि च स्युः । –

In all the above equations ⑥, ⑦ and ⑧ there are several unknowns in the numerator. If we give values for all of them, then 'a' may become a fraction or integer. When the solutions of the unknown should only be integral, then leaving one unknown, others should be given values. Then in the numerator we will have one unknown, others will be integers ; i.e. in the equation ⑥

Assuming $c=1 \quad d=2$, we have

$$a = \frac{5b-6-12}{2}$$

$$= \frac{5b-18}{2}$$

i.e. $$2a = 5b-18$$

This is obviously an equation which can be solved using *kuṭṭaka*. एवं कृते प्रथमवर्णमानमभिन्नमेव स्यात् । – This way 'a' the first unknown will definitely be an integer.

If, however, one fixes up integer values for three of the unknowns, the value we get for the fourth one may not be integral. *BP.* p. 212 : भाज्यवर्णमानानामनियतत्वात्तदिष्टकल्पनेन उद्दिष्टसिद्धिः स्यात् ।... यदि त्वनयोर्धनयोरन्यधनेनापि समतोद्दिष्टा स्यात्तदा भाज्यवर्णमानानामिष्टकल्पने व्यभिचारः स्यादेव । – Since there is no restriction as to the values given to the other unknowns, whatever values are assumed, one may be able to solve the equations. If however it is declared that these two wealths are equal to another wealth, too, then there will be contradiction if one assumes optional numbers for the unknowns in the dividend. If we assume $b=4$, $c=1$, $d=2$. Then from ⑥

$$a = \frac{5\times4-6\times1-6\times2}{2} = 1$$

Therefore we get the values $a=1$, $b=4$, $c=1$, $d=2$. On the other hand, if we assume $b=6$, $c=2$, $d=1$, then

$$a = \frac{5\times 5 - 6\times 2 - 6\times 1}{2}$$

$$= \frac{30 - 12 - 6}{2} = 6$$

Then we get the values $a = 6$, $b = 6$, $c = 2$ and $d = 1$.

But both these sets of values for b, c, d, do not give integral solution for a when they are substituted in equation ⑦. For instance, the first set of values gives

$$a = \frac{3b + c - d}{2} \qquad ⑦$$

$$= \frac{3\times 4 + 2\times 1 - 2\times 1}{2} = \frac{11}{2}$$

Since the example अश्वा: etc deals with horses, camels and so on, it is essential that the solutions of a, b, c and d should be positive, non-zero, integral and not fractional.

Working according to Bhāskara's *sūtra* (*BG.* vv. 152-53) :

आद्यं वर्णं शोधयेदन्यपक्षादन्यान्रूपाण्यन्यतश्चाद्यभक्ते ।
पक्षेऽन्यस्मिन्नाद्यवर्णोन्मितिः स्याद्वर्णस्यैकस्योन्मितीनां बहुत्वे ॥

समीकृतच्छेदगमे तु ताभ्यस्तदन्यवर्णोन्मितयः प्रसाध्याः ।
अन्त्योन्मितौ कुट्टविधेर्गुणाप्ती ते भाज्यतद्भाजकवर्णमाने ॥

– Dividing by the co-efficient of the first unknown, we get its *unmiti* (value). If we get more than one *unmiti* for one unknown, by equating them we get the values of other unknowns. We equate ⑥ and ⑦

$$\frac{5b - 6c - 6d}{2} = \frac{3b + c - d}{3}$$

since both are equal to a. Cross multiplying

$$15b - 18c - 18d = 6b + 2c - 2d$$

$$9b = 20c + 16d$$

$$b = \frac{20c + 16d}{9} \qquad ⑨$$

Similarly equating ⑦ and ⑧

$$\frac{3b + c - d}{3} = \frac{3b - 2c + d}{2}$$

$$6b + 2c - 2d = 9b - 6c + 3d$$

$$3b = 8c - 5d$$

$$b = \frac{8c - 5d}{3} \qquad ⑩$$

From ⑨ and ⑩ we have

$$\frac{20c + 16d}{9} = \frac{8c - 5d}{3}$$

$$60c + 48d = 72c - 45d$$

$$93d = 12c$$

$$31d = 4c$$

$$c = \frac{31}{4}d$$

For c to be an integer, d should be equal to $4t$, where t is any arbitrary number. Therefore $d = 4t$. If $d = 4t$, then $c = \frac{31}{4} \cdot 4t = 31t$. Substituting the value of c and d in equation ⑨

$$b = \frac{20c + 16d}{9}$$

$$= \frac{20 \times 31t + 16 \times 4t}{9}$$

$$= \frac{620t + 64t}{9}$$

$$= \frac{684t}{9} = 76t$$

Again substituting the values of b, c and d in ⑥

$$a = \frac{5b - 6c - 6d}{2}$$

$$= \frac{5 \times 76t - 6 \times 31t - 6 \times 4t}{2}$$

$$= \frac{380t - 186t - 24t}{2} = \frac{170t}{2} = 85t$$

We thus have $a = 85t$, $b = 76t$, $c = 31t$ and $d = 4t$. Giving value $t = 1$, then $a = 85$, $b = 76$, $c = 31$, $d = 4$ (*BP.* p. 215) : यद्येकमिष्टं कल्प्यते तर्हि जातानि यावत्तावदादिमानानि । 85, 76, 31, 4. If $t = 2$ then $a = 170$, $b = 152$, $c = 62$, $d = 8$ and so on.

Remarks : There are in this problem three linear homogeneous equations in four unknowns. The procedure adopted above for obtaining solutions is one of reduction to the case of two unknowns by eliminating the other two unknowns.

6.1.1. *Phalaikya śeṣaikya vicāra* :

The next example which Kṛṣṇa takes up for discussion is the following (*BG.* v. 163) :

नवभिः सप्तभिः क्षुण्णः को राशिस्त्रिंशता हृतः ।
यदग्रैक्यं फलैक्याढ्यं भवेत्षड्विंशतेर्मितम् ॥

– Find the number which multiplied by 9 and 7 separately and divided by 30 give such quotients and remainders that the sum of these four is 26.

Let the required number be x.

According to statement given

$$\frac{9x}{30} = Q_1 + \frac{R_1}{30} \quad ; \quad \frac{7x}{30} = Q_2 + \frac{R_2}{30}$$

i) where Q_1 and Q_2 are the quotients and R_1 and R_2 are the remainders.

ii) $Q_1 + Q_2 + R_1 + R_2 = 26$. In this case, $R_1 + R_2$ is less then 30, the divisor.
Adding the two equations we get

$$\frac{9x}{30} + \frac{7x}{30} = Q_1 + \frac{R_1}{30} + Q_2 + \frac{R_2}{30}$$

i.e.
$$\frac{16x}{30} = (Q_1 + Q_2) + \frac{R_1 + R_2}{30}$$

For example, when $x = 5$

$$\frac{9 \times 5}{30} = 1 + \frac{15}{30} \qquad Q_1 = 1 \quad R_1 = 15$$

$$\frac{7 \times 5}{30} = 1 + \frac{5}{30} \qquad Q_2 = 1 \quad R_2 = 5$$

so that

$$\frac{16 \times 5}{30} = 2 + \frac{20}{30}$$

Here $R_1 + R_2 = 20 < 30$. Let $Q_1 + Q_2 = Q$; $R_1 + R_2 = R$, then

$$\frac{16x}{30} = Q + \frac{R}{30}$$

so that

$$\begin{aligned} 16x &= 30Q + R \\ &= 30Q + (26 - Q) \\ &= 29Q + 26 \end{aligned}$$

We use *kuṭṭaka* to solve $16x = 29Q + 26$

Vallī

16) 29 (1	1	234	2	$x = 29 - 2 = 27$
16	1	130	2	$Q = 16 - 2 = 14$
13) 16 (1	4	104		
13	26			
3) 13 (4	0			
12				
1				

$Q = 16t + 14,\ x = 29t + 27,\ t = 0, \pm 1, \pm 2 \ldots$

We verify by substituting the value 27 for x corresponding to $t = 0$.

$$\frac{27 \times 9}{30} = \frac{243}{30} = 8 + \frac{3}{30}$$

$$\frac{27 \times 7}{30} = \frac{189}{30} = 6 + \frac{9}{30}$$

and $Q_1 + Q_2 = 8 + 6 = 14$; $R_1 + R_2 = 3 + 9 = 12$, so that $Q_1 + Q_2 + R_1 + R_2 = 26$

Remarks : The next value of x namely 56 does not satisfy the main requirement that $Q_1 + Q_2 + R_1 + R_2 = 26$, in this problem.

However, if $Q_1 + Q_2 + R_1 + R_2$ is greater than the divisor 30, (say 38) it may happen that $R_1 + R_2$ is greater than 30. Kṛṣṇa discusses this case (*BP.* pp. 220-22) : यद्यपि हरादधिकं शेषैक्ये सति । He examines, in this context, a case where $Q_1 + Q_2 + R_1 + R_2 = 38$ or $R + Q = 38$

$$\frac{16x}{30} = Q + \frac{R}{30}$$

$$16x = 30Q + R = 30Q + (38 - Q) = 29Q + 38$$

By *kuṭṭaka* we have $x = 29t + 6 \quad Q = 16t + 2,\ t = 0, \pm 1, \pm 2 \ldots$

Since $R + Q$ should add upto only 38, 't' can take the value 0, 1.

Let us take $t = 0$

$$\therefore \qquad x = 29(0) + 6 = 6$$

$$Q = 16(0) + 2 = 2$$

Now

$$\frac{9 \times 6}{30} = 1 + \frac{24}{30} = Q_1 + \frac{R_1}{30}$$

$$\frac{7\times 6}{30} = 1 + \frac{12}{30} = Q_2 + \frac{R_2}{30}$$

$$Q_1 + Q_2 = Q = 2$$

$$R_1 + R_2 = R = 36$$

$Q_1 + Q_2 + R_1 + R_2 = 38$ as desired. However this $Q_1 + Q_2$ is not the same as quotient obtained when $(9+7)6$ is divided by 30. Also $R_1 + R_2$ is not the same as this remainder, since

$$\frac{9\times 6 + 7\times 6}{30} = \frac{96}{30} = 3 + \frac{6}{30}$$

In the above example let Q' be the actual quotient, when $16x$ is divided by 30 and R' the remainder. Then R' is less than 30

$$\frac{16x}{30} = Q' + \frac{R'}{30}$$

$16x = 30\ Q' + R'$; $R' = 16x - 30Q'$. This new $Q' = Q_1 + Q_2 + 1$ and $R' = R_1 + R_2 - 30$ as observed by Kṛṣṇa[3] too : इदं फलं फलैक्यं सैकमस्तीति फलं रूपोनं सञ्जातं फलैक्यम् । Re-writing,

$$Q_1 + Q_2 = Q' - 1 \qquad ①$$

$$R_1 + R_2 = R' + 30$$

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3. Kṛṣṇa illustrates (*BP*. pp. 220-22) : For instance if $x = 6$

$$\frac{9\times 6}{30} = 1 + \frac{24}{30} \qquad Q_1 = 1 \qquad R_1 = 24$$

$$\frac{7\times 6}{30} = 1 + \frac{12}{30} \qquad Q_2 = 1 \qquad R_2 = 12$$

Here $R_1 + R_2 = 36$ is greater than divisor 30.

$$\frac{16\times 6}{30} = 3 + \frac{6}{30} = Q + \frac{R}{30}$$

$Q = Q_1 + Q_2 + 1$ (*BP*. p. 221) : पूर्वं फलैक्यं सैकमस्ति ।

$R = R_1 + R_2 -$ divisor. शेषं च शेषैक्यं हरतष्टमस्ति ।

Thus

$$R_1 + R_2 = 16x - 30Q' + 30 \qquad ②$$

and so (from ① and ②) $Q_1 + Q_2 + R_1 + R_2 = Q' - 1 + 16x - 30Q' + 30$

The requirement is now, in terms of Q',

$$38 = 16x - 29Q' + 29$$

or

$$16x = 29Q' + 9$$

We use *kuṭṭaka* :

	Vallī			
16) 29 (1	1	81	23	$x = 29 - 23 = 6$
16	1	45	13	$Q' = 16 - 13 = 3$
13) 16 (1	4	36		
13	9			
3) 13 (4	0			
12				
1				

Now, $$\frac{16 \times x}{30} = Q' + \frac{R'}{30}$$

i.e. $$\frac{16 \times 6}{30} = 3 + \frac{6}{30}$$

and $Q_1 + Q_2 + R_1 + R_2 = Q' - 1 + R' + 30 = 3 - 1 + 6 + 30 = 38$ as desired.

With the original equation, therefore the rule is if $R_1 + R_2$ is greater than the divisor take $Q = Q_1 + Q_2 + 1$ and $R = R_1 + R_2 -$ divisor and use the same method.

Though the above procedure is an alternative, in case the sum of the quotients and remainders is larger than the divisor, the following remarks,

in this context, made by Kṛṣṇa, discouraging the above procedure in favour of the earlier procedure valid for all values of $Q+R$ desired are noteworthy (*BP.* p. 222) :

इयांस्तु विशेषः । फलप्रमाणे कालके कल्पिते यदि फलैक्यशेषैक्ययोरन्यथात्वं निश्चितं स्यात् तर्हि एव फलं निरेकं शेषं च सहरं कर्तुं युज्यते नान्यथा । फलैक्ये तु कालके कल्पिते न कोऽपि विचारोऽस्तीति लाघवात्फलैक्यमेव कालकः कल्प्यत इति सर्वमवदातम् । – The special point to be remembered is this : If $Q_1 + Q_2 + R_1 + R_2$ is greater than the divisor then the last method can be considered remembering that $Q_1 + Q_2$ is $Q' -1$ and R' is $R_1 + R_2 -$ divisor. On the other hand by assuming the quotient Q to be $Q_1 + Q_2$ there will not be any hurdles and the correct solution will be obtained easily.

6.1.2. *Ṛṇa-śeṣa-labdhi vicāra* :

An example of previous authors (*ādya udāharaṇa*) is taken up by Bhāskara for discussion in this section. It has been discussed elaborately by Kṛṣṇa also. Bhāskara himself states that he has 'somehow' solved it. He has made his own views on the said problem quite clear in his commentary (*BG.* p. 99) :

. . . इदं अनियताधारक्रियायाम् आद्यैरुदाहृत्य यथाकथंचित् समीकरणं कृत्वाऽऽनीतम्। इयं तथा कल्पना कृता यथा अत्र अनियताधारायामपि नियताधारक्रियावत् फलम् आगच्छति । एवंविधकल्पनाच्च क्रियासंकोचाद् यत्र व्यभिचरति तत्र बुद्धिमद्भिः बुद्ध्या संधेयम् । – This is cited as an example of earlier authors with insufficient data and by framing equations, somehow a solution has been obtained. Here an assumption is made such that a solution transpires from insufficient data as from sufficient data. Even after this assumption if the procedure does not produce the answer, the intelligent (students) should find the right procedure using the intellect.[4]

4. *Cf.* tr. by H.T. Colebrooke, *op.cit.*, p. 243.

Note : Whenever there is insufficient data (the number of equations being less than the number of unknowns) the effort is to still find feasible solutions rather than leave the problem unsolved. In such a case, the 'Heuristic approach' – proposing hypotheses for the solutions and testing them out for practicality – can be used. Bhāskara and Kṛṣṇa seem to have adopted such a method in solving this problem.

This illustration is given below (*BG.* v. 168) :

षडष्टशतकाः क्रीत्वा समार्घेण[5] फलानि ये ।
विक्रीय च पुनः शेषमेकैकं पञ्चभिः पणैः ।।
जाताः समपणास्तेषां कः क्रयो विक्रयश्च कः ।।

– "Three traders, having six, eight, and a hundred, for their capitals respectively, bought fruits at an uniform rate ; and resold [a part] so ; and disposed of the remainder at one for five *paṇas* ; and thus became equally rich. What was [the rate of] their purchase? and what was [that of] their sale?"[6]

The assumptions made by Bhāskara before setting to solve the problem are as follows : 1) Let the number of fruits bought be x per *paṇa* ; 2) a is the amount for which they were sold for a *paṇa* in the first sale by the first trader ; 3) R_1, R_2, R_3 are the number of fruits remaining after the first sale by the three traders respectively ; 4) the fruits were sold by the traders at the rate of 110 per *paṇa* ; 5) the first sale by the three traders is in proportion to their wealth before sale, *viz.*, 6, 8, 100 which amounts to *trairāśika* $3 : 4 : 50 = 1 : \frac{4}{3} : \frac{50}{3}$. The last two assumptions are extraneous to the problem posed as such in the verse.

5. The variant reading समार्घेण, found in some of the editions of the *BP* seems to be corrupt or erroneous. Kṛṣṇa however, while solving the problem takes it for सममूल्येन meaning thereby 'at the same rate' – buying at a common rate and selling at a (different) common rate. Also, the *mulagrantha BG*, as edited by various scholars has the correct reading समार्घेण; *argha* means *mūlya* (rate).

13

6. Tr. H.T. Colebrooke, p. 242.

If z_1, z_2, z_3 are the amounts in rupees for which the three vendors respectively sold fruits in the first sale at the rate of l fruits per *paṇa* and R_1, R_2, R_3 are the fruits remaining thereafter correspondingly, we obtain the equations $6x = z_1 l + R_1$; $8x = z_2 l + R_2$; $100x = z_3 l + R_3$ and $z_1 + 5R_1 = z_2 + 5R_2 = z_3 + 5R_3$.

There are, in all 5 equations for 8 unknowns which can be reduced to three equations in 6 unknowns by eliminating two of R_1, R_2, R_3 from the last equation. No set procedure is feasible for solution of such equations where the unknowns are large in number compared to the number of equations for which, generally there will be infinite number of solutions. What Bhāskara and Kṛṣṇa do is to assume $l = 110$ and $z_1 : z_2 : z_3 = 6 : 8 : 100$ or $\frac{z_1}{6} = \frac{z_2}{8} = \frac{z_3}{100}$. Thus, they effectively, reduce the number of equations to 3, that too linear and still there are 5 unknowns.

Now we have $$6x = 110a + R_1$$

Verbally, the above equation means that the total number of fruits bought by the first trader is equal to the sum of those sold by him at 110 per *paṇa* in the first sale and the rest R_1 after this sale. So, $R_1 = 6x - 110a$ and R_1 is sold at 5 *paṇas* each. Hence

$$5R_1 = 30x - 550a$$

So the first man's wealth is

$$W_1 = a + 5R_1 = a + 30x - 550a = 30x - 549a \qquad ①$$

If we take 'a' as first man's first sale amount, then the second man's first sale will be, by *trairāśika* or proportion, $\frac{8}{6} \times a = \frac{4}{3}a$. Hence

$$\frac{8x}{110} = \frac{4a}{3} + \frac{R_2}{110}$$

where R_2 is the remainder of fruits after the second man's first sale which were sold at 5 *paṇa* each.

$$R_2 = \frac{24x - 440a}{3}$$

so that

$$5R_2 = \frac{120x - 2200a}{3}$$

Hence the second man's wealth

$$W_2 = \frac{4}{3}a + 5R_2 = \frac{120x - 2200a}{3} + \frac{4}{3}a$$

$$= \frac{120x - 2200a + 4a}{3} = \frac{120x - 2196a}{3} \quad (= 40x - 732a) \qquad ②$$

Since we have taken '*a*' as first man's first sale, the third man's first sale will, by *trairāśika* or proportion, be $\frac{100}{6} \times a = \frac{50a}{3}$

$$\frac{100x}{110} = \frac{50a}{3} + \frac{R_3}{110}$$

where R_3 is the remainder of fruits of third man's first sale which were sold at 5 *paṇa* each.

$$300x = 5500a + 3R_3$$

or
$$R_3 = \frac{300x - 5500a}{3}$$

i.e.
$$5R_3 = \frac{1500x - 27500a}{3}$$

Thus the third man's wealth

$$W_3 = \frac{50a}{3} + 5R_3 = \frac{50a}{3} + \frac{1500x - 27500a}{3}$$

$$= \frac{1500x - 27450a}{3} \quad (= 500x - 9150a) \qquad ③$$

According to the statement in the problem, the wealth of the three traders

are equal. Equating ① and ②

$$30x - 549a = \frac{120x - 2196a}{3}$$

i.e. $$90x - 1647a = 120x - 2196a$$

i.e. $$2196a - 1647a = 120x - 90x$$

i.e. $$549a = 30x$$

i.e. $$x = \frac{549a}{30}$$

Equating ② and ③

$$\frac{120x - 2196a}{3} = \frac{1500x - 27450a}{3}$$

i.e. $$1500x - 120x = 27450a - 2196a$$

i.e. $$1380x = 25234a$$

i.e. $$x = \frac{549a}{30}$$

The same solution $x = \frac{549a}{30}$ is obtained when the wealths of the first and third traders are equated. This is a *kuṭṭaka* whose solutions are $x = 549t + 0$ and $a = 30t + 0$. $t = 1, 2, 3 \ldots$ Putting $t = 1$, $x = 549$, $a = 30$; we therefore have the following result

First man's fruits	=	$549 \times 6 = 3294$
Second man's fruits	=	$549 \times 8 = 4392$
Third man's fruits	=	$549 \times 100 = 54900$

$\frac{3294}{110}$ gives $a = 29$, $R_1 = 104$

$\frac{4392}{110}$ gives $b = 39$, $R_2 = 102$

$\frac{54900}{110}$ gives $c = 499$, $R_3 = 10$

a, b, c being the quotients on division by 110. We have thus

First man's wealth W_1	$= a + R_1 \times 5$	$= 29 + 104 \times 5$	$= 549$
Second man's wealth W_2	$= b + R_2 \times 5$	$= 39 + 102 \times 5$	$= 549$
Third man's wealth W_3	$= c + R_3 \times 5$	$= 499 + 10 \times 5$	$= 549$

Note that $b \neq \frac{4}{3}a$, $c \neq \frac{50}{3}a$ if $a = 29$ as was assumed in the beginning as regards the first sale through *trairāśika*. Also the price x per *paṇa* in purchase and the amount a in the first sale by the first vendor, obtained as 549, 30 respectively are such that $6x = 3294 \neq 110 \times 30 + 104$. However $6x = 3294 = 110 \times 30 - 6 = 110 \times 30 - (110 - 104)$ seems to have motivated Kṛṣṇa on *ṛṇa-śeṣa-labdhi-vicāra.*

6.1.2.1. Questions raised and answered by Kṛṣṇa :

Two important questions arise and these are taken for discussion by Kṛṣṇa.

A) *BP.* p. 226 : अथात्र किंचित् विचार्यते । इह हि षड्गुणितात् क्रयात् विक्रयहृतात् यदि कालको लभ्यते तदाऽष्टगुणितात् शतगुणितात् च किमिति त्रैराशिकेन लब्धिसाधनं कृतमाचार्यैः । –If a is the first man's sale, using *trairāśika* can $\frac{4\dot{a}}{3} \left(= a \times \frac{8}{6}\right)$ be taken as the second man's sale?

B) *BP.* p. 228 : ननु अत्र यावत्तावत् उन्मानमिदं $\begin{matrix} \text{का} & ५४९ \\ \text{या} & ३० \end{matrix}$ ।

अत्र भाज्य-भाजकयोः त्रिभिः अपवर्तः संभवति । भाज्यो हारः क्षेपकश्चापवर्त्य इति कुट्टकार्थमावश्यकश्च सः । तत्कथं कृतेऽपवर्ते मानमसदागच्छति । – The evaluation of x, $\frac{549}{30}$ should be reduced to smaller numbers by dividing by 3, before doing *kuṭṭaka*. How do we not get the result in this case?

Note : In the ensuing discussion, writing the equation $6x = 110a + R_1$ in the form $\frac{6x}{110} = a + \frac{R_1}{110}$ is resorted to.

Question A : The question is 'if a is the quotient (*labdhi*) for $\frac{6x}{110}$, can $\left(a \times \frac{8}{6}\right)$ be the quotient for $\frac{8x}{110}$'.

For this we have to ask ourselves further,

a) षड्गुणितस्य क्रयस्य येहलब्धिः कल्पिता सा किमशेषा सशेषा वा । – Whether $\frac{6x}{110}$ has a remainder or is 110 an exact factor of 6x.

For this, Kṛṣṇa answers : आद्ये शेषाभावात् शेषमेकैकं पंचभिः पणैरित्यालापविरोधः । – If we assume $\frac{6x}{110}$ has no remainder, it goes against the statement that the remainder of fruits was sold for five *paṇa*. Also if $R_1 = R_2 = R_3 = 0$, then the wealth of the three traders after sale turn out to be $a, \frac{4a}{3}, \frac{50a}{3}$ which are unequal unless $a = 0$. In other words $R_1, R_2, R_3 \neq 0$. So, $\frac{6x}{110}$ is not an integer.

द्वितीये तु तादृशलब्धेरनुपातेन गुणान्तरलब्धिसाधनमयुक्तम् । – Also, the two quotients do not have the relationship for *trairāśika* to work. Kṛṣṇa proves this as below :

Let us take the example 15 ÷ 13. If we multiply 15 by 6 and divide by 13, we have

$$\frac{90}{13} = Q_1 + \frac{R_1}{13} \qquad = \qquad 6 + \frac{12}{13} \quad Q_1 = 6 \ ; \ R_1 = 12$$

If we multiply 15 by 8 and divide by 13 we have

$$\frac{120}{13} = Q_2 + \frac{R_2}{13} \qquad = \qquad 9 + \frac{3}{13} \quad Q_2 = 9 \ ; \ R_1 = 3$$

Obviously *trairāśika* does not work here since it is seen that

$$Q_2(9) \neq \frac{4}{3}\ (Q_1) \neq \frac{4}{3} \times 6$$

b) The rate of sale at the first instance (chosen as 110 by Bhāskara) (इष्टविक्रय) should be more than any of the multipliers – केवलभाज्यस्य हि खण्डद्वयमस्ति । यावद्धरभक्तं तावदेकम् । शेषतुल्यमपरम् । प्रथमखण्डं केवलमपि हरभक्तं शुद्ध्यतीति गुणकेन गुणितं तत्सुतरां शुद्ध्येत् । तस्य लब्धिस्तु केवलभाज्यस्य या लब्धिः सैव गुणकगुणिता स्यात् । अतस्तत्रानुपातो युक्त एव । – *Kevalabhājya* or pure dividend (the quantity divided) has two parts ; the part divided by the divisor < without remainder > and remainder. Even after multiplication by a multiplier, the quotient multiplied by the factor will be divided clearly (i.e. without remainder). The new quotient *may be* the product of the old quotient and the multiplier. Therefore taking the new quotient as a multiple of the old one, is only correct.

Now, taking the second part and multiplying it by the multiplier, we get the product of the two. If this product is less then the divisor, this product will itself be the remainder. If this product is greater than the divisor, then it will have a quotient and remainder. एवं केवलभाज्ये हरेण भक्ते यदि रूपं शेषं स्यात्तदा गुणितभाज्यस्य द्वितीयखण्डं गुणतुल्यमेव स्यादिति गुणाधिके हरे शेषोथ्थलब्धेः अभावाल्लब्ध्यनुपातो युक्त एव । – Thus when the remainder is one, like for example $15 \div 14$ giving $Q = 1$, $R = 1$. When we multiply 15 by 6 which is less than 14, the new quotient will be 6 and new remainder is also 6, which is the multiplier. अत एवाचार्यैर्गुणाधिक एव इष्टविक्रयः कल्पितः । – That is the reason why Ācārya has taken *iṣṭavikraya* to be 110 which is greater than the multipliers 6, 8 or 100. यदि तु गुणान्न्यून इष्टविक्रयः कल्प्येत तदाऽनुपातलब्धौ व्यभिचारः स्यात् । – if the assumed *iṣṭavikraya* is less than 6, 8 or 100 it would not help to solve the problem in the example given. We may also encounter another difficulty : यस्य गुणकस्य लब्धिरल्पा तस्य शेषमप्यल्पं यस्य च लब्धिरधिका तस्य शेषमप्यधिकं स्यादिति पणसाम्यं कथमपि न स्यात् । – when the quotient is small, the remainder would be small ; when the quotient is big, the remainder would be big. This way the wealths of the three persons cannot be equal.

Number 110 (for the rate of sale in the first sale) is chosen larger than 6, 8, 100, for the reason that the larger the number of fruits with a trader, it is expected for equalization of wealth, the smaller the number of fruits remain for the second sale by 5 *paṇas* each. Otherwise the second sale will help traders with more initial wealth to earn more wealth in the second sale. With this thinking in mind, again, *trairāśika* is resorted to. The more the initial wealth the more the number of fruits sold proportionately in the first sale.

c) Another justification attempted by Kṛṣṇa for use of *trairāśika* is that Bhāskara (possibly) considered negative complementary remainder (*ṛṇaśeṣa*) instead of the positive remainder (*dhanaśeṣa*). In this case, he asserts that numerically the negative complementary remainder is small when the quotient is large. Hence, *dhanaśeṣalabdhi* is taken as *ṛṇaśeṣalabdhi* less one – Kṛṣṇa explains further (*BP*. p. 227) : सशेषा लब्धिस्तावत् द्विविधा । धनशेषा ऋणशेषा चेति । शेषमपि द्विविधं धनमृणं चेति । – Quotients with remainders can be of two kinds, *viz.,* those with positive remainder and those with negative remainder.

A numerical example will explain this. Suppose 29 is divided by 13. 29 ÷ 13 gives 2 as quotient (*labdhi*) and 3 and remainder (*śeṣa*)

i.e. $$\frac{29}{13} = 2 + \frac{3}{13}$$

Here since 3 is positive, it is called *dhanaśeṣa* and 2 is called *dhanaśeṣa-labdhi.*

$\frac{29}{13}$ can also be written as $3 - \frac{10}{13}$. Here since 10 is negative, it is called *ṛṇaśeṣa* and 3 is *ṛṇaśeṣalabdhi.*

अत: धनशेषा लब्धि: सैका सती ऋणशेषा लब्धि: स्यात् । इयं वा निरेका सति धनशेषा लब्धि: स्यात् । एव धनर्णशेषयोगो हरतुल्योऽस्तीति धनशेषं हराच्छोधितं सदृणशेषं स्यात् । इदं

हराच्छोधितं सद्धनशेषं स्यात् । – It is obvious that the difference between the two *labdhis* is one. In the above example, 3 – 2 = 1 ; similarly the numerical sum of the two *śeṣa* will be the divisor itself. i.e. 3 + 10 = 13. Therefore to make *ṛṇaśeṣa* into *dhanaśeṣa,* the *ṛṇaśeṣa* should be subtracted from the divisor.

BP. p. 228 : अत्र शेषाणि ऋणं सन्तीति धनत्वार्थं तानि हराच्छोध्यानि । तथासति गुणकोनहर: शेषं स्यादिति यस्य गुणकस्य लब्धिरधिका तस्य शेषमल्पम् । यस्य लब्धिरल्पा तस्य शेषमधिकं स्यादिति पणसाम्यं संभवेत् । – When the remainder is negative, to make it positive it has to be subtracted from the divisor. So, if the quotient of the multiplier is big, the remainder is small and vice versa. Therefore the wealth can be equated.

अत आचार्यै: ऋणशेषा लब्धि: कालकमिता कल्पिताऽस्तीति न कोपि दोष: । – For this same reason, the assumption of Bhāskara that '*a*' or quotient is *ṛṇaśeṣalabdhi* is acceptable.

ऋणशेषा लब्धयो निरेका: सत्यो धनलब्धय: स्युरिति अनुपातजलब्धीर्निरेका: कृत्वा कर्म कर्तुं युज्यते । आचार्यैस्तु न तथा कृतमिति कथं दोषो न स्यादिति चेत् न । – Also since the difference between *ṛṇaśeṣalabdhi* and *dhanaśeṣalabdhi* is one, in the given example the quotients should be assumed after subtracting one (1) from *ṛṇaśeṣa*. If you say, Ācārya has not done this, Kṛṣṇa says it is not a fault : यतस्तथाकरणे पक्षेषु समान्येव रूपाण्यधिकानि स्यु: अकरणे तु रूपाभाव एवेति आचार्यकृतपक्षास्तुल्यैरेव रूपैरूना जाता इति ते साम्यं न त्यजन्तीति । – Since the same number 1 is subtracted, the equations are not affected in any way and therefore equating the two sides is acceptable. This is how Kṛṣṇa defends Bhāskara's use of *trairāśika* in the example.

Question B : Now the other question asked is : should not $30x = 549a$ be reduced to smaller numbers by dividing by 3, before doing *kuṭṭaka* ?

After equating equations ① and ② or ② and ③ we have

$$30x = 549a \quad \text{or} \quad x = \frac{549}{30}a$$

Thus *kuṭṭaka* is done getting $x = 549t$, $a = 30t$. According to the *sūtra* on *kuṭṭaka* (*BP.* p. 228) : 'भाज्यो हारः क्षेपकश्चापवर्त्यः केनाप्यादौ संभवे कुट्टकार्थम् '[7]। कृतेऽपवर्ते मानमसदागच्छति । अनपवर्ते च सत् । इह हि शेषमावश्यकम् । कृतेत्वपवर्ते शेषाण्यपवर्तितानि स्युरिति नोद्दिष्टसिद्धिः । – both 549 and 30 should be reduced by dividing by 3 before performing *kuṭṭaka.* But by doing so, one does not get the correct solution. By not reducing, the correct solution is obtained. In this example it is essential to keep the remainders as they are. By reduction, these also get reduced.

Kṛṣṇa (*BP.* p. 228) : explains his *guru's* statement by quoting Bhāskara's own verse from *Golapraśnādhyāya* :

उद्दिष्टं कुट्टके तज्ज्ञैर्ज्ञेयं निरपवर्तनम् ।
व्यभिचारः क्वचित्क्वापि खिलत्वापत्तिरन्यथा ॥

The broad meaning is that *nirapavartana* or 'no reduction' should be done if it will lead to an incorrect answer.

Note : Kamalākara (17th Cent. A.D.) does not agree with this view of Kṛṣṇa. In his *Siddhāntatattvaviveka,*[8] Kamalākara states (Mahāpraśnādhikāra, v. 255) :

नवाङ्कुरे बीजोत्थे कुट्टकानपवर्तने ।
सिद्धान्तसंमतिर्योक्ता सदर्थाज्ञानतोऽस्ति सा ॥

– The words quoted in *Navāṅkura* (=*BP*) that 'the said *anapavartana* has sanction from *Siddhānta* (*Śiromaṇi*) ' is only from ignorance.

Mm. Sudhakara Dvivedi in his edition of the *BG* takes objection to the method propounded by Bhāskara. He says (*BG.* p.94) : आचार्योक्त्या च

7. *BG.* v. 56.

8. *Siddhāntatattvaviveka* of Kamalākara Bhaṭṭa, ed., with notes Mm. Sudhakara Dvivedi and Mm. Muralidhara Jha, Chaukhamba Surabharati Prakasan, Varanasi, 1991.

कुट्टकविधे "येनच्छिन्नौ भाज्यहारौ न तेन क्षेपः चैतत् दुष्टमुद्दिष्टमेव" इत्यनेन नावासर इत्याचार्यकल्पना मन्दानन्दकरीति गणितरसिकज्ञैर्निपुणं विलोकनीयम् । – Bhāskara has given the *sūtra* on *kuṭṭaka* stressing the need for *apavartana.* To say that 'here it is not applicable' is only satisfying the duller intellects.

In this connection, he also adds in a different context[9] that – षडष्टशतकाः क्रीत्वेत्याद्युदाहरणे बीजांङ्कुरायां " उद्दिष्टं कुट्टके तज्ज्ञैर्ज्ञेयो निरपवर्तनम्" इति भास्करोक्त गोलाध्यायस्थ वचनम् अज्ञात्वैव सर्वं कृष्णदैवज्ञेन यदुक्तं तद्बुद्धिमद्भिर्निपुणं विलोक्यम् ।। – While explaining the example *ṣaḍaṣṭaśatakāḥ,* whatever Kṛṣṇa has said without understanding the real import of Bhāskara's statement given in the *Golapraśnadhyāya,* is to be looked into by mathematicians.

In the same edition, in his expository notes, Mm. Muralidhara Jha does not concur with Mm. Sudhakara Dvivedi. He gives the following rejoinder – आलापवत् कृत्वा समीकरणात् **या = का** $\frac{549}{30}$ । अत्र हरभाज्यौ त्रिभिर्नापवर्तितौ इष्टराशेरलब्धत्वात् । एतत् सर्वं ज्ञात्वैव श्रीमद् भास्कराचार्येणोक्तम् - "एवंविधकल्पनात् क्रियासंकोचात् यत्र व्यभिचरति तत्र बुद्धिमद्भिर्बुद्ध्या संधेयम्" । एवं स्पष्टं स्वदोषं स्वीकृत्य आचार्यस्य कल्पना मन्दानन्दकरीति न रोचते । – Bhāskara has not done *apavartana* knowing fully that by doing so, a solution will not be possible (in this context). But Ācārya has also added that in certain expectional cases, such assumptions should be made by the wise. So it is not correct to use the phrase 'pleasing the duller intellects'.

Remarks : As regards reduction of the *kuṭṭaka* $30x = 549a$ to the lowest form, *viz.,* $10x = 183a$, which yields the solution $x = 183t$, $a = 10t$, $t = 1,2,3,\ldots$, the convention is to choose the value of t which is relevant to the problem. But whatever be t the solution amounts to each vendor having zero wealth after sale. For, $W_1 = 30x - 549a$, $W_2 = 40x - 732a$, $W_3 = 500x - 9150a$ which vanish whether $10x = 183a$ or, equivalently $30x = 549a$. The conclusion in the solution that $W_1 = W_2 = W = 549$ has

9. *Gaṇaka Taraṅgiṇī, op.cit.,* p. 70

been arrived under the premise that R_i , $i = 1,2,3$, are remainders on division of the number of fruits held by the respective traders before sale by 110 (the rate of first sale). The money for which they sold in the first sale is the quotient in this division. Thus, taking the *kuṭṭaka* in a particular form is not an issue of any relevance since the solution overlooks the expression for the final wealth of the three traders already obtained to get a solutions. Note that the final 'a' $\neq a$, $b \neq 4/3a$, $c \neq 50/3a$, while one choice 0of a has already been shown to be equal to 30, corresponding to $x = 549$. Thus $W_1 = 30 \times 549 - 549 \times 30 = 0$ and likewise W_2, $W_3 = 0$.

However, it is to be noted that one solution has been obtained by making intelligent working assumptions for the problem of the verse stated *viz.*, $x = 549$, $z_1 = 29$, $z_2 = 39$, $z_3 = 499$, $R_1 = 104$, $R_2 = 102$, $R_3 = 10$. The remarks of Bhāskara made with some reservation and quoted in the beginning of the section **6.1.2.** are certainly meaningful.

Also, it is to be noted, that on verifying backwards from the solution $x = 549$, $a = 30$ and $R_l = 104$, the original equation $6x = 110a + R_1$ seems to be a misconception.

6.1.2.2. Kṛṣṇa's solution :

Kṛṣṇa gives 'two solutions' to the same problem discussed above in **6.1.2.** to avoid doubts about the validity raised earlier.

In addition to the assumption made by Bhāskara keeping *iṣṭavikraya* as 110 (greater than any of the multipliers 6, 8, 100), Kṛṣṇa also assumes that (*BP.* p. 228) *kevalakraya* or fruits bought per *paṇa* to be more than the *iṣṭavikraya* 110. (*BP.* p. 230) : इहाधिकगुणात् शतादेकगुणादेव विक्रयोऽधिकोस्तीति केवलक्रयस्य रूपमेव ऋणशेषं संभवति नान्यत् । – Since 110 (*iṣṭavikraya*) divided by the largest multiplier *viz.*, 100 gives quotient 1, we

can safely assume *ṛṇaśeṣa* to be 1 only. Any number greater than 1 would make the remainder more than divisor. For instance

$$\frac{x}{110} = Q - \frac{1}{110}$$ (keeping *ṛṇaśeṣa* as 1). Then

$$\frac{100 \times x}{110} = 100Q - \frac{100}{110}$$. On the other hand if *ṛṇaśeṣa* = 2, then

$$\frac{x}{110} = Q - \frac{2}{110}$$

$$\frac{100 \times x}{110} = 100Q - \frac{200}{110}$$

In this case, we cannot use *ṛṇaśeṣalabdhi* since 200 > 110.

अतो जातं व्यक्तमेव केवलक्रयस्य ऋणशेषं १̇ । – Therefore *ṛṇaśeṣa* is assumed to be −1.

<u>Method 1 :</u> Let the number of fruits bought for a *paṇa* be *x*, the number of fruits sold (*iṣṭavikraya*) for a *paṇa* is assumed to be 110 which is greater than 6, 8 or 100. Let *y* be the *ṛṇaśeṣalabdhi* (quotient). Then

$$\frac{6x}{110} = 6y + \textit{ṛṇaśeṣa}$$

To convert *ṛṇaśeṣalabdhi* to *dhanaśeṣalabdhi* , the number one has to be subtracted from *ṛṇaśeṣalabdhi*. Only then can we have a positive remainder as given in the problem (*BP.* p. 229) : एता निरेका जाता धनशेषा लब्धयः । – By subtracting 1 *ṛṇaśeṣalabdhi* becomes *dhanaśeṣalabdhi*.

$$\frac{6x}{110} = 6y - 1 + \frac{R_1}{110}$$

Similarly $$\frac{8x}{110} = 8y - 1 + \frac{R_2}{110}$$

and $$\frac{100x}{110} = 100y - 1 + \frac{R_3}{100}$$

Therefore

$$\begin{aligned} 6x &= 660y - 110 + R_1 \\ 8x &= 880y - 110 + R_2 \\ 100x &= 11000y - 110 + R_3 \end{aligned}$$

So

$$\begin{aligned} R_1 &= 6x - 660y + 110 \\ R_2 &= 8x - 880y + 110 \\ R_3 &= 100x - 11000y + 110 \end{aligned}$$

Since the remainder of fruits were sold for 5 *paṇa* each,

$$\begin{aligned} 5R_1 &= 30x - 3300y + 550 \\ 5R_2 &= 40x - 4400y + 550 \\ 5R_3 &= 500x - 55000y + 550 \end{aligned}$$

Therefore first man's wealth after sale is equal to

$$6y - 1 + 30x - 3300y + 550$$

$$= 30x - 3294y + 549 \quad ①$$

Second man's wealth after sale is equal to

$$8y - 1 + 40x - 4400y + 550$$

$$= 40x - 4329y + 549 \quad ②$$

Third man's wealth after sale is equal to

$$100y - 1 + 500x - 55000y + 550$$

$$= 500x - 54900y + 549 \quad ③$$

Since the wealth of the three are equal, equating ① and ②

$$30x - 3294y + 549 = 40x - 4392y + 549$$

$$10x = 1098y$$

$$5x = 549y, \quad x = \frac{549}{5}y$$

By *kuṭṭaka* we have

$$x = 549t + 0$$

$$y = 5t + 0, \qquad t = 0, \pm 1, \pm 2, \pm 3 \ldots$$

Here Kṛṣṇa makes a valid remark (*BP.* p. 229) : अत्र नीलकमेकेनैवोत्थापयेत् । अन्यथा क्रये विक्रयेण हृते रूपाधिकमृणशेषं स्यादिति शेषोत्थलब्धि संभवेन लब्धिव्यभिचारात् मानमसत् स्यात् । – Here also 't' cannot be more than the value 1. Any other value for 't' will result in *ṛṇaśeṣa* being greater than 1 and quotient would be more than remainder, which is absurd. Thus we have

$$x = 549, \quad y = 5.$$

$$R_1 = 6x - 660y + 110 = 6 \times 549 - 660 \times 5 + 110 = 104$$
$$R_2 = 8x - 880y + 110 = 8 \times 549 - 880 \times 5 + 110 = 102$$
$$R_3 = 100x - 11000y + 110 = 100 \times 549 - 11000 \times 5 + 110 = 10$$

First man's wealth after sale $= 5R_1 + 6y - 1$

$$= 5 \times 104 + 6 \times 5 - 1 = 520 + 29 = 549$$

Second man's wealth after sale $= 5R_2 + 8y - 1$

$$= 5 \times 102 + 8 \times 5 - 1 = 510 + 39 = 549$$

Third man's wealth after sale $= 5R_3 + 100y - 1$

$$= 5 \times 10 + 100 \times 5 - 1 = 50 + 499 = 549$$

Method 2 : Kṛṣṇa gives yet another method for the same problem. Here the assumption is $\frac{x}{110} = Q - \frac{1}{110}$.

$$\therefore \quad \frac{6x}{110} = 6Q - \frac{6}{110} = 6Q - 1 + \frac{104}{110}$$

$$\frac{8x}{110} = 8Q - \frac{8}{110} = 8Q - 1 + \frac{102}{110}$$

$$\frac{100x}{110} = 100Q - \frac{100}{110} = 100Q - 1 + \frac{10}{110}$$

Kṛṣṇa proceeds to explain (*BP.* p. 230) : In the first case, since 6 is *ṛṇaśeṣa*, and since the sum of *ṛṇaśeṣa* and *dhanaśeṣa* is the divisor, the *dhanaśeṣa* would be 110 – 6 = 104. Similarly in the second case, it would be 110 – 8 = 102 ; and in the third, it would be 110 –100 = 10. Since these remainders are sold for 5 *paṇa* each, the wealth out of these would be 520, 510 and 50 respectively.

Wealth of first person after sale $= 6Q - 1 + 520 = 6Q + 519$

Wealth of second person after sale $= 8Q - 1 + 510 = 8Q + 509$

Wealth of third person after sale $= 100Q - 1 + 50 = 100Q + 49$

Since the wealth of the three persons are equal,

$$\begin{aligned} 6Q + 519 &= 8Q + 509 \\ 2Q &= 10 \\ Q &= 5 \end{aligned}$$

The same result will be obtained by equating the second and third equations or the third and first equations. The three quotients are : $6Q - 1$, $8Q - 1$, $100Q - 1$ i.e. 29, 39 and 499 respectively. Since $\frac{x}{110} = Q - \frac{1}{110} = 5 - \frac{1}{110}$ or $\left(4 + \frac{109}{110}\right) = \frac{549}{110}$. So,

$$x = 549$$

Thus in both the above methods, the two questions (A, B) mentioned before have been answered.

Remarks : The basic equation in Bhāskara's solution *viz.*, $6x = 110a + R_1$ is now rewritten as

$$6x = 110 \times 6y - R_1'$$

and in the equivalent form $$\frac{6x}{110} = 6y - \frac{R_1'}{110} \quad ①$$

when y is defined explicitly in Method 1 as *ṛṇaśeṣalabdhi*. Again, R_1 is the negative complementary remainder when $6x$ is divided by 110. In otherwords, a the amount for which the first sale was made by the first trader is written in the form $6y$. In the context of second and third vendors the corresponding equations are written respectively as

$$\frac{8x}{110} = 8y - \frac{R_2'}{110}$$

$$\frac{100x}{110} = 100y - \frac{R_3'}{110}$$

R_2', R_3' being negative complementary remainders in numerical value in the division of $8x, 100x$ respectively by 110. First, *trairāśika* is implicit in the formation of the equations. Secondly ① can be rewritten as

$$\frac{6x}{110} = 6y - 1 + \frac{R_1}{110} \qquad ②$$

where R_1 is the actual remainder when $6x$ is divided by 110 (and so less than 110). Reverting to the form in which this could be interpreted in the context of the given problem

$$6x = (6y - 1)110 + R_1$$

Thus a in Bhāskara's solution is replaced by $6y - 1$ and R' is replaced by R_1 where R_1 is the remainder in $6x$ when divided by 110, the number of fruits sold in the second sale by the first vendor. With this, however, unlike in Bhāskara's solution (where the wealth are all 0), the wealth of all the three treaders are equal to 549 *paṇas*.

6.2. Kṛṣṇa's own Examples :

Here Kṛṣṇa adds a couple of examples of his own (*BP.* p. 231) :

<u>Example 1:</u> सप्ताऽऽसन् मणिवणिजोऽत्र योऽधिकश्रीः स प्रादात् परधनसंमितं परेभ्यः।
प्रत्येकं परसममेवमेव दत्वा ये जाताः समपणयोगं किं धनास्ते ॥

14.

– There were seven traders. The wealthier of them gave wealth equal to that of others. Each, after giving wealth equal to others had equal *paṇas.* What were the initial wealth?

This problem being vaguely stated would require several variables. However Kṛṣṇa reformulates the problem to give a method by which it could be solved using a single variable. He says : अत्र मणिप्रमाणानि यावत्तावदादीनि प्रकल्प्य अनेकवर्णसमीकरणेन साध्यानि । अस्यानयनार्थं व्यक्तरीत्या मत्सूत्रमप्यस्ति । – This problem can be solved in the method of *anekavarṇasamīkaraṇa* ; however to solve it in a simpler way, the following *sūtra* is given. The *sūtra* is :

वद सैकनरैर्मितमेकधनं द्विगुणं विधुहीनमिदं तु परम् ।
अमुना विधिना परतोऽपि परं द्विगुणं द्विगुणं द्वयमेव समम् ।।

– Each man's wealth is twice the previous man's wealth less 1. By this rule, find each man's wealth and the distribution by which their wealth is made equal.

Remarks : The problem posed in the second verse does not seem to have any bearing to the problem posed in the first verse. The method also does not use any of the data given in the first verse except the fact that there were seven traders. With this basic premise that there were seven traders, Kṛṣṇa solves the problem using the above rule in the following manner.

Let first man's wealth be		x
Then second man's wealth	$=$	$2x-1$
Third man's wealth	$=$	$2(2x-1)-1$
	$=$	$4x-3$
Fourth man's wealth	$=$	$2(4x-3)-1$
	$=$	$8x-7$

Fifth man's wealth	=	$2(8x-7)-1$
	=	$16x-15$
Sixth man's wealth	=	$2(16x-15)-1$
	=	$32x-31$
Seventh man's wealth	=	$2(32x-31)-1$
	=	$64x-63$
Total	=	$127x-120$

For all the seven people to have equal wealth, the total (= $127x - 120$) should be exactly divisible by 7 under the assumption that even wealth cannot be in fractions.

i.e. $$127x - 120 = 7y \text{ (say)}$$

By *kuṭṭaka* we have solution sets (1,1) (8, 128) . . . Discarding (1,1) which gives equal wealth of 1 for each, (8, 128) gives the least solution. Therefore the wealth of the seven people are 8, 15, 29, 57, 113, 225 and 449.

Sum of the wealth = 896

and each one's share $= \frac{896}{7} = 128$

6.2.1. General solution for the example given by Kṛṣṇa :

Removing the restriction that there were 7 persons, we solve the problem for any number of persons.

Case 1 : - The wealth of the successor is 2 times the wealth of the predecessor less 1, and there are n persons. The wealth are in the sequence

$$x,\ 2x-1,\ 4x-3,\ 8x-7\ \ldots$$

i.e. $$x,\ 2\times x-1,\ 2(2x-1)-1,\ 2(4x-3)-1,\ \ldots$$

The sequence can be re-written as

$$2^{k-1}x-(2^{k-1}-1),\quad k=1,2,3\ldots n$$

According to the problem, the sum of these n expressions divided by n (the number of persons) is y, which is the amount received by each person. Now the sum of x, $2x-1$, $4x-3$ to n terms is

$$\sum_{i=1}^{n}2^{i-1}x \quad - \quad \sum_{i=1}^{n}(2^{i-1}-1)$$

$$\therefore \quad = x\sum_{i=1}^{n}2^{i-1} - \sum_{i=1}^{n}2^{i-1}+n$$

$$= x(2^n-1)-(2^n-1)+n \qquad \left\{\because \sum_{i=1}^{n}2^{i-1}=2^n-1\right\}$$

The above expression gives the sum to n terms which is the amount available for equal distribution. Hence

$$\frac{x(2^n-1)-(2^n-1)+n}{n} = y$$

$$\Rightarrow \quad x(2^n-1)-(2^n-1)+n = ny$$

$$\Rightarrow \quad x(2^n-1)-(2^n-1) = ny-n$$

$$\Rightarrow \quad (2^n-1)(x-1) = n(y-1)$$

$$\Rightarrow \quad \frac{(x-1)}{n} = \frac{y-1}{2^n-1} = k\ \text{(say)}$$

i.e. $$\frac{(x-1)}{n} = k\ ,\quad \frac{y-1}{2^n-1} = k$$

$$x-1=nk\ ,\ \text{i.e.}\ x=nk+1,\quad y=k(2^n-1)+1$$

Putting $k=1$,

$$x=n+1\ ,\quad y=2^n$$

Putting $n = 7$ in above we have $x = 8$, $y = 2^7 = 128$, the wealth are 8, 15 . . . 449. Adding, the sum of wealth is 896 and dividing by 7, $\frac{896}{7} = 128 = 2^7$

Case 2 : Here we will discuss the same problem when the wealth of the successor is m times the wealth of the predecessor less 1. In other words, the amount the n persons have are x, $mx - 1$, $m(mx - 1) - 1$, $m\{m\,(mx - 1) - 1\} - 1$, . . . , i.e. x, $mx - 1$, $m^2x - m - 1$, $m^3x - m^2 - m - 1$, . . .

$$i^{\text{th}} \text{ term} = m^{i-1}x - \frac{m^{i-1} - 1}{m - 1}$$

$$\text{Sum to } n \text{ terms} = \sum_{i=1}^{n} m^{i-1}x - \sum_{i=1}^{n} \frac{m^{i-1} - 1}{m - 1}$$

$$= \sum_{i=1}^{n} m^{i-1}x - \frac{1}{m-1}\sum_{i=1}^{n} m^{i-1} + \sum_{i=1}^{n} \frac{1}{m-1}$$

$$= \frac{m^n - 1}{m - 1}x - \frac{1}{(m-1)}\frac{(m^n - 1)}{(m-1)} + \frac{n}{m-1}$$

Since this sum is divided equally amongst n people and each gets y.

$$\frac{\frac{m^n - 1}{m - 1}x - \frac{1}{(m-1)}\frac{m^n - 1}{(m-1)} + \frac{n}{m-1}}{n} = y$$

$$\frac{m^n - 1}{m - 1}x - \frac{m^n - 1}{(m-1)^2} + \frac{n}{m-1} = ny$$

Multiplying throughout by $(m - 1)^2$

$$(m^n - 1)(m - 1)x - (m^n - 1) + n(m - 1) = ny(m - 1)^2$$

$$(m^n - 1)\;(x(m - 1) - 1) = ny(m - 1)^2 - n(m - 1)$$

$$(m^n - 1)\;\{x(m - 1) - 1\} = n(m - 1)\{y(m - 1) - 1\}$$

$$\Rightarrow \quad \frac{x(m - 1) - 1}{n(m - 1)} = \frac{y(m - 1) - 1}{m^n - 1} = k \text{ (say)}$$

So

$$x(m-1)-1 = kn(m-1)$$

$$x = \frac{kn(m-1)+1}{m-1}, \quad y = \frac{k(m^n-i)+1}{m-1}$$

When $m = 2$ and $k = 1$ we obtain the earlier case.

$$x = \frac{kn(m-1)+1}{m-1} = n+1$$

$$y = \frac{k(m^n-1)+1}{m-1} = 2^n$$

Remark: Prof. T. Hayashi indicates that the procedure is really solving the recurrent relation

$x_1 = n + 1$

$x_j = 2x_{i-1} - 1, (i \geq 2)$

Solution for common wealth = 2^n

<u>Example 2</u> : श्रीकृष्णेन यदिन्द्रनीलपटलं क्रीतं प्रियार्थं ततो
भागं भीष्मसुताऽष्टमं यदधिकं रूपं तदप्याददे ।
सत्याद्या: पुनरेवमेव विदधु: सम्प्राप्यनालोकिता:
पत्यु: प्रापुरिमा: पुन: समलवं सानन्दमादिं वद ।।

– a heap of *Indranīla* gems were bought by Śrī Kṛṣṇa for his wives. He gave $\frac{1}{8}$ of the same along with remaining one gem to Rukmiṇī. In the same manner to Satyabhāmā and the other six wives, he gave $\frac{1}{8}$ of the remainder plus one gem, in that order. At the end all were happy with the equal share (*samalava*). Find the total number of gems that Śrī Kṛṣṇa bought in the beginning.

Note : The author Kṛṣṇa has worked out the problem partially.

The problem can be formulated thus :

$$x = 8f_i + 1$$

$$7f_i = 8f_{i+1} + 1, \ (i = 1,2, \dots 7)$$

$$7f_8 = 8f_9$$

So far, a procedure to solve such an equation is not known.

6.3. Viṣṇu Daivajña's methods discussed by Kṛṣṇa :

1) Continuing the discussion, on the example *ṣaḍaṣṭaśatakāḥ . . .*, Kṛṣṇa adds a *sūtra* by his revered *guru* Viṣṇu Daivajña which gives another method for solving the problem (*BP.* p. 231) :

शेषविक्रयहृतेष्टविक्रयः शीतरश्मिरहितो भवेत्क्रयः ।
पुंधनादधिक इष्टविक्रयः कल्प्य इत्थमवगम्य धीमता ।।

– *Śeṣavikraya* (amount at which remainder is sold) multiplied by *iṣṭavikraya* (number of fruits per *paṇa* sold in the first sale) less one[10] (1) gives the purchase rate, when *iṣṭavikraya* is assumed to be more than any of the multipliers.

Kṛṣṇa explains (*BP.* p. 231) : एकस्य शेषफलस्य विक्रयलभ्याः पणाः इह शेषविक्रयो विवक्षितः । स चात्र पंच । – that by *śeṣavikraya* is meant the money given for sale of the remainder of fruits i.e. 5 *paṇas.* According to Viṣṇu Daivajña

क्रयः = शेषविक्रय (= 5) × इष्टविक्रय (= 110) – शीतरश्मि (= 1) = 549

i.e. *kraya* (sale) is arrived at by multiplying *śeṣavikraya* and *iṣṭavikraya* and subtracting 1 from the product. The condition given is that *iṣṭavikraya*

10. The word *śītaraśmi* means moon ; being a *bhūtasaṅkhyā* it indicates number 1.

should be assumed to be greater than the *pumdhana* (original money with each) namely six, eight or hundred.

Note : This is not a solution, but just a rule to remember the final solution.

Since Viṣṇu Daivajña's rule does not involve any of the multipliers 6, 8 or 100 the answer for *kraya* will be the same for all the three vendors.

Proof: इष्टविक्रय × शेषविक्रय − 1

$= 110 \times 5 - 1$

$= (104 \times 5) + (6 \times 5) - 1$

$= (104 \times 5) + (30) - 1$

$= (104 \times 5) + 29 + 1 - 1$

$= 520 + 29 = 549$

or $110 \times 5 - 1$

$= (102 \times 5) + (8 \times 5) - 1$

$= (102 \times 5) + (40) - 1$

$= (102 \times 5) + 39 + 1 - 1$

$= 510 + 39 = 549$

2) In the context of discussing the example *eko bravīti* . . . **(5.3.1)**, Kṛṣṇa gives Viṣṇu Daivajña's simple method :

एको ब्रवीति मम देहि शतं धनेन त्वत्तो भवामि हि सखे द्विगुणस्ततोऽन्यः ।
ब्रूते दशार्पयसि चेन्मम षड्गुणोऽहं त्वत्तस्तयोर्वद धने मम किं प्रमाणे ।।

"One man says to the other, 'Please give me Rs.100, then my wealth will be twice your wealth'. The other says, 'If you will give

me Rs.10, my wealth will be six times your wealth'. Find each man's wealth''. [11]

Viṣṇu Daivajña's method (*BP.* p. 216) :

स्वस्वैकयुक्तगुणदानजघातयोर्योऽनल्पः परः परगुणाभिहतस्तदैक्यम् ।
तत्स्यान्निरेकगुणघातहृतं हि राशिस्तत्संगुणाधिकगुणः परवर्जितः सन् ॥
द्वितीयराशिमानं स्यादव्यक्तक्रियया विना ।
व्यक्तमव्यक्तयुक्तं यद्येन बुद्ध्यन्ति ते जडाः ॥

– In general, let the wealth of the two people be x and y respectively, the second man tells the first, 'If you give me rupees '*c*', I shall be '*a*' times as rich as you.' The first man says, 'If you give me rupees '*d*', I shall be '*b*' times as rich as you.' Then $x = \frac{(ac+c)b+bd+d}{ab-1}$ and $y = \frac{abd+ad+ac+c}{ab-1}$.
The two equations relating to the general problem would be :

$$a(x-c) = y+c \quad ①$$
$$x+d = b(y-d) \quad ②$$

11. In the Chapter on *Ekavarṇa samīkaraṇa* the same example had been solved using a single variable. Here it is solved using two variables.
According to the above two statements let x and y be the wealth of first and second man respectively. Then,

$$2(x-100) = (y+100) \quad ①$$
and
$$x+10 = 6(y-10) \quad ②$$

① can be written as

$$2x-200 = y+100$$
i.e.
$$2x-y = 300 \quad ③$$

② can be written as

$$x+10 = 6y-60$$
i.e.
$$-x+6y = 70 \quad ④$$
③ × 6
$$12x-6y = 1800 \quad ⑤$$
④ + ⑤
$$11x = 1870 \Rightarrow x = 170$$
from ③
$$2x-300 = y$$
$$2\times170-300 = 340-300 = 40 = y$$

$$ax - ac = y + c$$

$$x + d = by - bd$$

$$ac + c = ax - y \quad ③$$

$$bd + d = -x + by \quad ④$$

To solve, multiply ④ by b

$$abc + bc = abx - by \quad ⑤$$

$$bd + d = -x + by \quad ④$$

Adding ⑤ + ④

$$abc + bc + bd + d = (ab - 1)x$$

$$x = \frac{(ac + c)b + bd + d}{ab - 1}$$

$$ac + c = ax - y \quad ③$$

$$bd + d = -x + by \quad ④$$

Multiplying ④ by a

$$abd + ad = -ax + aby \quad ⑥$$

$$ac + c = ax - y \quad ③$$

Adding ⑥ and ③

$$abd + ad + ac + c = (ab - 1)y$$

$$y = \frac{abd + ad + ac + c}{ab - 1}$$

Actually Viṣṇu Daivajña's formula reduces to :

$$x = \frac{(ac + c)b + bd + d}{ab - 1}$$

Kṛṣṇa explains (*BP*.p.216) – अथ प्रथम: गुण: ($=a$) दानं च ($=c$) द्वितीयो गुण: ($=b$) दानं च ($=d$) एकयुक्तगुणेन स्वस्वदाने गुणिते जातौ स्वस्वैकयुक्तगुणदानजघातौ ($= ac + c$) and ($= bd + d$) ; let ($= ac + c$) be the greater of the two. अयमन्यस्य गुणेन 'b' गुणित: $b\,(ac + c)$; द्वितीयस्तु यथास्थित एव ($bd + d$). अनयोरैक्यं $(ac + c)b + (bd + d)$. इदं गुणघातेन निरेकेन ($= ab - 1$). हृतं जातो राशि: । Therefore,

$$x = \frac{(ac + c)b + (bd + d)}{ab - 1}$$

In the example $a = 2$; $b = 6$; $c = 100$; $d = 10$ so that

$$x = \frac{(2 \times 100 + 100)6 + (6 \times 10 + 10)}{12 - 1} = \frac{1870}{11} = 170$$

अनेन $\left[= \frac{\{(ac + c)b + (bd + d)\}}{ab - 1} \right]$ अधिकस्य गुणे ($=a$) गुणित $\left[= \frac{\{(ac + c)b + (bd + d)\}a}{ab - 1} \right]$ परेणानेन ($= ac + c$) वर्जितो जातो द्वितीयराशिरिति । i.e.

$$y = \frac{\{(ac + c)b + (bd + d)\}a}{(ab - 1)} - (ac + c)$$

$$= \frac{(abc + bc + bd + d)a}{ab - 1} - (ac + c)$$

$$= \frac{a^2bc + abc + abd + ad - (ac + c)(ab - 1)}{(ab - 1)}$$

$$= \frac{a^2bc + abc + abd + ad - a^2bc - abc + c + ac}{ab - 1}$$

$$= \frac{abd + ad + ac + c}{ab - 1}$$

In the example $a = 2$; $b = 6$; $c = 100$; $d = 10$, so that

$$y = \frac{2 \times 6 \times 10 + 2 \times 10 + 2 \times 100 + 100}{2 \times 6 - 1}$$

$$= \frac{120 + 20 + 200 + 100}{11}$$

$$= \frac{440}{11} = 40$$

Note : Viṣṇu Daivajña has given only the rule to express a solution; but not the working expressed above.

6.4. *Anekavarṇa samīkaraṇa Madhyamāharaṇa bheda* :

Bhāskara is perhaps the earliest author to mention the solution of indeterminate equations of the second degree other than *varga-prakṛti* in his *BG.* But since he himself has taken examples from other authors[12], it is highly probable that the method was known to algebraists before him. However Datta and Singh observe[13] : "Neither those illustrations nor a treatment of equations of these types occur in the algebra of Brahmagupta or in any other extant work anterior to Bhāskara II".

According to Bhāskara (*BG.* p. 106): एवं तदैव यदा असकृत् समीकरणं यदा तु सकृदेव समीकरणं . . . । – there are two kinds of these indeterminate equations *sakṛt samīkaraṇa* (simple equation) and *asakṛt samīkaraṇa* (multiple equation)

6.4.1. Simple Equations :

Under this head, a few types of such equations are discussed.

6.4.1.1. Equations of the type $ax^2 + bx + c = y^2$:

BG. gives the following *sūtras* (vv. 171-72) :

एकस्य पक्षस्य पदे गृहीते द्वितीयपक्षे यदि रूपयुक्तः ।
अव्यक्तवर्गोऽत्र कृतिप्रकृत्या साध्ये तदा ज्येष्ठकनिष्ठमूले ।।
ज्येष्ठं तयोः प्रथमपक्षपदेन तुल्यं कृत्वोक्तवत्प्रथमवर्णमितिः प्रसाध्या ।
ह्रस्वं भवेत्प्रकृतिवर्णमितिः सुधीभिरेवं कृतिप्रकृतिरत्र नियोजनीया ।।

– When the square root of one side is taken, if there is the square of unknown accompained by absolute number on the other side, by the method of solving *varga-prakṛti* we should have *jyeṣṭha* and *kaniṣṭha*. Equating the *jyeṣṭha* to the square root of the first side, we get the value of x. And the *hrasva* should be taken as the value of the co-efficient of *prakṛti* .

12. For instance, before giving the example "यत्स्यात् . . . विना" (*BG*. v.185) Bhāskara says : "अथ कस्याप्युदाहरणम्" implying that he is quoting some other author whose name is not known to us.
13. Datta and Singh, *op.cit.*, Vol. II. p.181.

The example given by Bhāskara is (*BG.* v. 173) :

को राशिर्द्विगुणो राशिवर्गैः षड्भिः समन्वितः ।
मूलदो जायते बीजगणितज्ञ वदाऽऽशु तम् ॥

– What number when doubled and added to six times its square, becomes capable of yielding a square root? O algebraist, tell me quickly.

Kṛṣṇa gives the solution of the above problem in detail : To solve the equation for x and y in :

$$6x^2 + 2x = y^2$$

Multiplying by 6 throughout,

$$36x^2 + 12x = 6y^2$$

Adding 1 to both sides

$$36x^2 + 12x + 1 = 6y^2 + 1$$

$$(6x+1)^2 = 6y^2 + 1$$

Let $6x + 1 = X$

$$X^2 = 6y^2 + 1$$

This is obviously a case of *varga-prakṛti*

The solution sets are (2,5), (20,49), . . . For example,

$$X = 5 \qquad y = 2$$

$$X = 6x + 1 = 5$$

$$6x = 4$$

i.e. $$x = \frac{4}{6} = \frac{2}{3}$$

Using the above method of Bhāskara, infinite number of solutions can be obtained. This easy method has also been followed by later alegbraists like Nārāyaṇa and Jñānarāja, according to Datta and Singh. They add that some of these methods were rediscovered in 1733 by Euler and that : "His (Euler's) method is indirect and cumbrous. Lagrange's (18th Cent. A.Γ

method begins in the same way as that of Bhāskara II by completing the square on the left-hand side of the equation''.[14]

6.4.1.2. Equation of the type $ax^4 + bx^2 = y^2$:

Kṛṣṇa raises the doubt that (*BP.* p. 236) : अथ यदि द्वितीयपक्षे साव्यक्त-वर्गोऽव्यक्तवर्गवर्ग: स्यात् तदा नासौ वर्गप्रकृतेर्विषय : । – if the equation has powers of 4 and 2 (i.e. x^4, x^2) then using the method of solving *varga-prakṛti* may not be possible, he answers with another rule given by Bhāskara (*BG.* v. 175) :

द्वितीयपक्षे सति संभवे तु कृत्याऽपवर्त्यात्र पदे प्रसाध्ये ।
ज्येष्ठं कनिष्ठेन तथा निहन्याच्चेद्वर्गवर्गेण कृतोऽपवर्त: ।।
कनिष्ठवर्गेण तदा निहन्याज्ज्येष्ठं तत: पूर्ववदेव शेषम् ।।

Kṛṣṇa illustrates the above rule with the following example (*BG.* v. 176) :

यस्य वर्गकृति: पञ्चगुणा वर्गशतोनिता ।
मूलदा जायते राशिं गणितज्ञ वदाशु तम् ।।

''Find the value of x from $5x^4 - 100x^2 = y^2$ ''.

$$5x^4 - 100x^2 = y^2$$

$$\cdot \quad x^2(5x^2 - 100) = y^2$$

Kṛṣṇa says (*BP.* p. 237) – द्वितीयपक्षेऽव्यक्तवर्गवर्गोऽव्यक्तवर्गश्च स्यात्तदा अव्यक्तवर्गेणापवर्ते कृते सरूपोऽव्यक्तवयर्ग: स्यात् । ... न हि वर्गराशिर्वर्गेण गुणितो भक्तो वा वर्गत्वं जहाति । – Even after dividing by x^2 the left side still remains a square. i.e. $z^2 = \dfrac{y^2}{x^2}$.

By the method of of solving *varga-prakṛti* the solution for $5x^2 - 100 = z^2$ is found out. Solution sets are (10,20) (170,380), . . .

14. *ibid.*, p. 186.

$$z = \sqrt{5x^2 - 100}$$

Now

$$y^2 = x^2 (5x^2 - 100)$$

i.e. $$y = x\sqrt{5x^2 - 100}$$

$$= xz = 10 \times 20 = 200$$

is one of the solutions, for instance.

अत्र जातो यो वर्ग: स पूर्वोक्तयुक्त्या ज्येष्ठवर्ग एव परमेतस्य पदं न पूर्वपक्षपदसमम् । – The *jyeṣṭhamūla* 'z' that is obtained by solving *varga-prakṛti* is evidently not the initial *mūla* 'y'. अव्यक्तस्य तु मानं व्यक्तमेव कनिष्ठरूपं जातमस्ति । – The *kaniṣṭhamūla* 'x' is of course unchanged. अत: कनिष्ठवर्गेण गुणितो ज्येष्ठवर्ग: पूर्वपक्षसम: स्यात् – i.e. $y^2 = x^2 z^2$. अस्य पदं तु कनिष्ठगुणितं ज्येष्ठमेव । $y = x$ multiplied by the *jyeṣṭha* 'z'. अत उपपन्नं ''ज्येष्ठं कनिष्ठेन तदा निहन्यात्'' इति । – Thus it is proved that the *kaniṣṭhamūla* should be multiplied by the *jyeṣṭhamūla.* एवं वर्गवर्गेण अपवर्ते कृते ज्येष्ठवर्ग: प्रथमपक्षसाम्यार्थं कनिष्ठवर्गवर्गेण गुणनीय: । तस्य च पदं कनिष्ठवर्गगुणितं ज्येष्ठमेव । – Similarly if the equation has been divided by a biquadratic factor, the *jyeṣṭhamūla* should be multiplied by the square of the *kaniṣṭhamūla* to get the solution. अत उपपन्नं चेत् वर्गेण कृतोऽपवर्त: कनिष्ठवर्गेण तदा निहन्याज्ज्येष्ठमिति । – Therefore it is said, ''if it is practicable, divide by the square of smaller root to obtain the greater root''.

Note : *Kaniṣṭha* and *jyeṣṭha* are terms which depend on the equation for solution.

6.4.1.3. Equations of the type $ax^2 + bx + c = a'y^2 + b'y + c'$:

Kṛṣṇa now takes up equations of the above type and discusses in detail (*BP.* pp. 237-38) : अथ यत्रैकस्य पक्षस्य पदे गृहीते सति द्वितीयपक्षे साव्यक्तोऽव्यक्तवर्ग: सरूपो अरूपो वा भवति तदाऽसौ वर्गप्रकृतेर्न विषय: – Since both sides are quadratics, it is not possible to keep one side as a square with

constant term, and the other side as another square without constant term in order to solve *varga-prakṛti,* as such. Therefore Bhāskara gives another *sūtra* (*BG.* v. 178) :

साव्यक्तरूपो यदि वर्णवर्गस्तदाऽन्यवर्णस्य कृतेः समं तम् ।
कृत्वा पदं तस्य तदन्यपक्षे वर्गप्रकृत्योक्तवदेव मूले ।।
कनिष्ठमाद्येन पदेन तुल्यं ज्येष्ठं द्वितीयेन समं विदध्यात् ।।

- If it is possible to find the square root of one side and the second side contains unknown, that side may be assumed to be square of some unknown. The square root of the first side should be found and the square root of the other side should be found by solving the *varga-prakṛti*. That will be *jyeṣṭhamūla* for the *varga-prakṛti* . The *kaniṣṭhamūla* may be equated with the square root first found. From this we can get the value of the first unknown ; from the *jyeṣṭhamūla,* we get the value of second unknown.

The following example illustrates equations of the type mentioned above (*BG.* v. 179) :

त्रिकादिद्व्युत्तरश्रेढ्यां गच्छे क्वापि च यत्फलम् ।
तदेव त्रिगुणं कस्मिन्नन्यगच्छे भवेद्वद ।।

– The first term of an arithmetic progression is 3 and the common difference is 2. Three times the sum of the first x terms is equal to the sum of the first y terms. Find x, y.

The sum of the first x terms is $\frac{x}{2}\left\{2 \times 3 + (x-1)2\right\} = x(x+2)$.

So $\quad y^2 + 2y \;=\; 3(x^2 + 2x) \;=\; 3x^2 + 6x$

अथ चतुराहतवर्गसमैरित्यादिना द्वितीयपक्षेऽव्यक्तवर्गो अव्यक्तं रूपाणि च तथा स्युर्यथामूलं लभ्येत । The square root of the l.h.s. is found using Śrīdhara's rule (see **5.5.1**) for a quadratic equation.

Multiplying by 3 and adding 9 to both sides,

$$3y^2 + 6y + 9 = 9x^2 + 18x + 9$$

$$3y^2 + 6y + 9 = (3x + 3)^2$$

Assume $$z^2 = 3y^2 + 6y + 9$$

Then $$z^2 = 3y^2 + 6y + 9 = (3x + 3)^2$$

एकस्य पक्षस्य पदे गृहीते सति यो द्वितीयपक्षे साव्यक्तोऽव्यक्तवर्गः सरूपोऽरूपो वा स्यात्स वर्गराशिरेव । अत उक्तं तदान्यवर्णस्य कृते समं तमिति। अत्र द्वितीयपक्षस्य प्रथमपक्षेणापि साम्यमस्ति कल्पिततृतीयवर्णवर्गेणापि साम्यमस्तीति प्रथमपक्षस्य तृतीयवर्णवर्गसाम्यं बलात् भाव्यम् । – [Denoting $(3x + 3)^2$ as side 1, $3y^2 + 6y + 9$ as side 2 and z as side 3] equating side 2 to side 1, side 2 becomes a square. Since these two sides are equal, by equating side 2 with side 3, side 3 becomes equal to side 1.

$$3y^2 + 6y + 9 = z^2$$

$$3y^2 + 6y = z^2 - 9$$

$$9y^2 + 18y = 3z^2 - 27$$

$$9y^2 + 18y + 9 = 3z^2 - 18$$

$$(3y + 3)^2 = 3z^2 - 18$$

तृतीये तु सरूपोव्यक्तवर्गः स्यादित्ययं वर्गप्रकृतेः विषयः – since the r.h.s. has a square term and constant term it is a case for *varga-prakṛti.* Now,

$$3z^2 - 18 = (3y + 3)^2$$

Let $$(3y + 3)^2 = w^2$$

Thus $$3z^2 - 18 = w^2$$

By the method of solving *varga-prakṛti*, *kaniṣṭhamūla* $z = 9$, *jyeṣṭhamūla* $w = 15$.

(*BP.* p. 239) – अतः तृतीयपक्षस्य ज्येष्ठवर्णात्मकस्य यत्पदं ज्येष्ठस्वरूपं तत्

द्वितीयपक्षपदेनैव समं भवितुमर्हति न प्रथमपक्षपदेन । अत उपपन्नं द्वितीयेन समं विदध्यादिति । – The *jyeṣṭhamūla* (w) obtained from this equation should be equated to side 2 (the term containing y). अथ तृतीय पक्षे वर्गप्रकृत्या पदे गृह्यमाणे यत्कनिष्ठं तदेव प्रागुक्तयुक्त्या तृतीयवर्णमानम् । तच्च प्रथमपक्षपदेन तुल्यं भवितुमर्हति . . . अत उक्तं "कनिष्ठमाद्येन पदेन तुल्यम्" इति । – Similiarly the *kaniṣṭhamūla* obtained from this equation is the thirdside as said before. This should be equated to square root of side 1.

$$z = 9 \quad ; \quad w = 15 ;$$

$$3y + 3 = 15$$

$$3y = 12 \quad ; \quad y = 4$$

Since $$(3x+3)^2 = 3y^2 + 6y + 9 = z^2 ,$$

$$3x + 3 = z = 9$$

$$3x = 6 \quad , \quad x = 2$$

Therefore the number of terms in the first series $x = 2$ and the number of terms in the second series $y = 4$.

Since there can be infinite solutions for *varga-prakṛti*, both 'x' and 'y' have infinite values.

6.4.2. Multiple Equations :

"Multiple equations" are those where three or more functions, linear or quadratic of the unknown variables, have to be made squares or cubes in order to solve the equation. There may or may not be a *kṣepa* in the equation.

These types of equations are found in the *Laghubhāskarīya* of Bhāskara I and *Br.Sp.* of Brahmagupta. Nārāyaṇa who is later to Bhāskara II also deals with such equations[15].

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15. *ibid.*, p. 284.

6.4.2.1. Equation of the type $ax^2 + by^2 + c = z^2$

$$a'x^2 + b'y^2 + c' = u^2$$

The solutions of the above type of equation is outlined by the *sūtra* (*BG.* v. 180) :

सरूपके[16] वर्णकृती तु यत्र तत्रेच्छयैकां प्रकृतिं प्रकल्प्य ।
शेषं ततः क्षेपकमुक्तवच्च मूले विदध्यादसकृत्समत्वे ॥

– Where on one side we have squares of two unknowns and a constant term, we should regard the co-efficient of one unknown as *prakṛti* and the rest as *kṣepa* and get the roots by the process of solving the *varga-prakṛti*. After that the two sides should be equated.

For illustrating the *sūtra* Kṛṣṇa takes the following example of Bhāskara (*BG.* v. 181) :

तौ राशी वद यत्कृत्योः सप्ताष्टगुणयोर्युतिः ।
मूलदा स्याद्वियोगस्तु मूलदो रूपसंयुतः ॥

– Tell me two numbers whose squares multiplied by 7 and 8 and added together is a square. The difference between these two added to one is also a square.

The problem reduces to solving the equations :

$$7x^2 + 8y^2 = z^2$$
$$7x^2 - 8y^2 + 1 = w^2$$

16. Jivanatha Jha, *op.cit.*, p. 527 : Though the *sūtra* says सरूपके (unknowns with integers) the example given by Bhāskara तौ राशी. . . ।, does not have any integers on the left or right side. Therefore in his *Siddhāntasundara*, Jñānarāja has changed the word from *sarūpaka* to *arūpaka*. But Kamalākara, in his *Siddhāntatattvaviveka* argues that Bhāskara's original word was *sarūpaka* only and gives his own illustration. आचार्येण सरूपके वर्णकृती तु यत्रेत्यादिना यदुक्तं तस्योदाहरणं न दर्शितं तौ राशी वद यत्कृत्यो इत्युदाहरणे पूर्वपक्षे वर्गाङ्कावेव रूपाणि न सन्ति अत एव सिद्धान्तसुन्दरकारेण ज्ञानराजेन निजकृतबीजे अरूपके वर्णकृती तु यत्रेत्यादिनोक्तं परं भास्कराचार्योक्तमूलसूत्रं तथाविधं नास्ति ततस्तदभिमतमुदाहरणं सिद्धान्ततत्त्वविवेके प्रश्नाध्याये सार्द्धेन भुजङ्गप्रयातेन दर्शितम् ।

However, Bhāskara himself has explained in his commentary that the rule is applicable to both *sarūpaka* and *arūpaka* (*BG.* p. 106) : यत्र प्रथमपक्षमूले गृहीते द्वितीयपक्षे वर्णयोः कृती सरूपे अरूपे वा भवतस्तत्रैकां वर्णकृतिं प्रकृतिं प्रकल्प्य शेषं क्षेपम् ।

Kṛṣṇa explains that there should not be any difficulty even if there is a *kṣepa* in the example. According to the *sūtra*, the first unknown is kept as *prakṛti* and the remaining terms, as *kṣepa*. He says : यदि रूपाणि भवेयुः तर्हि तान्यपि क्षेपपक्षे प्रकल्प्यानि । – Here the remaining term would consist of the second unknown and *kṣepa* 1.

(*BP.* p. 240) – अत्र "इष्टं ह्रस्वं"[17] इत्यादिकरणे कनिष्ठं व्यक्तं कल्पनीयम् । क्षेपजातीयो वर्णः कनिष्ठं कल्प्यं यतस्तथा सति तस्य वर्गः प्रकृत्य गुणितः क्षेपसजातीयो वर्णवर्गः स्यादित्युभयोः सजात्यात् योगे सति वर्णवर्ग एव स्यादतोऽस्य पदं संभवेत् ।– 'x' is assumed to be some '$m \times y$' so that the left side takes the form of a *varga-prakṛti*.

Let $x = 2y$ in the first equation

$$\begin{aligned} 7(2y)^2 + 8y^2 &= z^2 \\ 28y^2 + 8y^2 &= z^2 \\ 36y^2 &= z^2 \\ 6y &= z \end{aligned}$$

Therefore x and z are obtained in terms of y i.e. $x = 2y$; $z = 6y$. Substitute for x in second equation

$$\begin{aligned} 7(4y^2) - 8y^2 + 1 &= w^2 \\ 28y^2 - 8y^2 + 1 &= w^2 \\ 20y^2 + 1 &= w^2 \end{aligned}$$

By solving the *varga-prakṛti* equation $20y^2 + 1 = w^2$, the solution sets are (2,9) (36,161) . . .

$$y = 2 \quad w = 9$$

$$y = 36 \quad w = 161$$

when $y = 2$, $x = 2y = 4$; $z = 6y = 12$

when $y = 36$, $x = 2y = 72$; $z = 6y = 216$

Remark : The equation is solved only under the constraint $x = 2y$.

17. *BG.* v. 78.

6.4.2.2. Equation of the form $ax^2 + bxy + cy^2 = z^2$:

Kṛṣṇa explains the above equation thus (*BP.* p. 241) : यत्रैकस्य पक्षस्य पदे गृहीते द्वितीयपक्षे यदि वर्णवर्गो भावितं च स्यात् । – When the equation contains squares of unknowns and their product on one side and a square of unknown on the other, then Bhāskara (*BG.* v. 183) gives the rule :

सभाविते वर्णकृती तु यत्र तन्मूलमादाय तु शेषकस्य ।
इष्टोद्धृतस्येष्टविवर्जितस्य दलेन तुल्यं हि तदेव कार्यम् ॥

– If on the one side of an equation, there are squares of two unknowns with their product, we should extract a square root and the remainder may be divided by a desired number and then decreased by that number. After the difference is halved it may be equated with the square root.

The example given to illustrate the above is (*BG.* v. 184) :

ययोर्वर्गयुतिर्घातयुता मूलप्रदा भवेत् ।
तन्मूलगुणितो योगः सरूपश्चाऽऽशु तौ वद ॥

– Find two unknowns such that sum of their squares added to their product is a square. The square root multiplied by the sum of the above numbers added to one gives a square.

In algebraic notation, let the two numbers x and y be such that

$$x^2 + y^2 + xy = z^2 \quad ①$$

$$z(x + y) + 1 = w^2 \quad ②$$

Multiplying the first equation by 36,

$$36x^2 + 36xy + 36y^2 = 36z^2 = (6z)^2 = z_1^2 \text{ , say}$$

where $z_1 = 6z$. Now,

$$36x^2 + 36xy + 9y^2 + 27y^2 = z_1^2$$

i.e. $36x^2 + 36xy + 9y^2 + 27y^2 = (6x + 3y)^2 + 27y^2 = z_1^2$ ③

$(6x+3y)^2$ can be treated as *labdhi* or quotient and $27y^2$ as *śeṣa* or remainder. According to the above rule, divide *śeṣa* $27y^2$ by any number say y and subtract the same number and divide by 2.

$$\frac{\frac{27y^2}{y} - y}{2} = 13y$$

Let
$$13y = 6x+3y$$
$$10y = 6x$$
$$x = \frac{5}{3}y$$

Therefore $\frac{5}{3}y$, y are the two numbers. Substituting in equation ①

$$\left(\frac{5y}{3}\right)^2 + y^2 + \frac{5y}{3}y = \frac{25y^2}{9} + y^2 + \frac{5y^2}{3}$$

$$= \frac{25y^2+9y^2+15y^2}{9} = \frac{49y^2}{9} = \left(\frac{7y}{3}\right)^2$$

a square. Substituting in equation ②

$$\frac{7y}{3}\left(\frac{5y}{3}+y\right)+1 = \frac{7y \times 8y}{9} + 1 = \frac{56y^2}{9} + 1 = w^2$$

or
$$56y^2 + 9 = 9w^2$$

This *varga-prakṛti* equation can be solved in the normal manner. On inspection, the auxiliary equation is $56(2)^2 + 1 = 225 = 15^2$. So

$$56(2)^2(3)^2 + (3)^2 = 15^2(3)^2$$

i.e.
$$56(6)^2 + 9 = 9(15^2)$$

i.e.
$$y = 6 \quad , \quad w = 15$$

is a solution. If $y = 6$, then $x = \frac{5}{3} \times y = \frac{5}{3} \times 6 = 10$. The two numbers are 10, 6. Since we can obtain infinite solutions for *varga-prakṛti*, we can have many solutions for x and y.

Kṛṣṇa's *Upapatti*:

(*BP.* p. 241) : एकस्य पक्षस्य पदे गृहीते सति यो द्वितीयपक्षः स भावितवर्ण-वर्गद्वयात्मकोऽस्ति स वर्ग एव । अथ यावतस्तत्खण्डस्य मूलं लभ्यते तत् खण्डमपि वर्गराशिरेव । – since $x^2 + xy + y^2 = z^2$ a square, the left side is also a square. Even if we split the terms on the left side it will still be a square. By equation ③

$$(6x + 3y)^2 + 27y^2 = 36z^2 = z_1^2$$

$$27y^2 = z_1^2 - (6x + 3y)^2 = \{z_1 - (6x + 3y)\}\{z_1 + (6x + 3y)\}$$

अतोन्तरमिष्टं प्रकल्प्य वर्गान्तरं राशिवियोगभक्तमित्यादिना योगः स्यात् । – then the difference of the squares divided by the difference of the numbers gives the sum of the numbers. Put

$$w = z - 3\sqrt{3}\,y$$

since

$$(z - 3\sqrt{3}\,y)\,(z + 3\sqrt{3}\,y) = z^2 - 27y^2$$

$$\frac{z^2 - 27y^2}{w} = z + 3\sqrt{3}\,y$$

(*BP.* p. 241) : अथाभ्यां योगान्तराभ्यां "योगोऽन्तरेणोनयुतोर्द्धितश्च"[18] इति संक्रमणेन राशी स्याताम् । – Having got their sum and difference, the two numbers can be obtained by *saṅkramaṇa.*

$$z + 3\sqrt{3}\,y = \frac{z^2 - 27y^2}{w}$$

$$z - 3\sqrt{3}\,y = w$$

अथ योगोऽन्तरयुतोऽर्द्धितश्च बृहद्राशिः स्यात् । – By the method of *saṅkramaṇa* adding the two and dividing by 2 we get z.

$$z = \frac{\frac{z^2 - 27y^2}{w} + w}{2}$$

एवं योगोऽन्तरेण विवर्जितोस्य दलं लघुराशिः स्यात् । – Subtracting the second from the first equation and dividng by $2 \times 3\sqrt{3}$ we get y

18. *Līlāvatī*, v.61.

$$y = \frac{\dfrac{z^2 - 27y^2}{w} - w}{2 \times 3\sqrt{3}}$$

अत इष्टकल्पितेन अन्तरेण विवर्जितस्यास्य यद्दलं स लघुराशिरिति जातम् । ... । अत उपपन्नं शेषकस्य इष्टोद्धृतस्येष्टविवर्जितस्य दलेन तुल्यं हि तदेव कार्यम् इति । – Therefore dividing the remainder by any chosen number and subtracting the same number and dividing the result by 2 gives the solution.

Thus, from the above discussions, it is seen that Bhāskara and later Kṛṣṇa have given solutions of more general equations of the second degree. Bhāskara with remarkable ingenuity reduces them to the form $Nx^2 + 1 = y^2$. Mostly these methods indicate rational solutions ; but they also hint at integral solutions in special problems.

6.5. Linear Multiple Equations :

Regarding linear multiple equations, Kṛṣṇa says (*BP.* p. 242) : अत्रालापानां बहुत्वे असकृत्क्रिया निर्वहति अतो बुद्धिमता तथा राशी कल्प्यौ यथा एकेनैव वर्णेन सर्वेप्यालापा घटेरन् । – since this is a case of multiple equations, the wise should assume a single variable which satsifies all the statements.

Bhāskara's *sūtras* pertaining to linear multiple equations is (*BG.* vv. 187-88) :

सरूपमव्यक्तमरूपकं वा वियोगमूलं प्रथमं प्रकल्प्यम् ।
योगान्तरक्षेपकभाजिताद्यद्वर्गान्तरक्षेपकतः पदं स्यात् ।।
तेनाधिकं तत्तु वियोगमूलं स्याद्योगमूलं तु तयोस्तु वर्गौ ।
स्वक्षेपकोनौ हि वियोगयोगौ स्यातां ततः संक्रमणेन राशी ।।

– First of all we can think of a new unknown with or without a number as the square root of the difference of two unknowns. After that we divide the *kṣepa* corresponding to the difference of squares of the given unknowns by the *kṣepa* of the sum and difference of these unknowns and find the

square root of the quotient. When that root is added to the root of *viyogamūla* as assumed before we get the *yogamūla.* Then the *yogamūla* and *viyogamūla* are squared and from them the respective *kṣepa* are subtracted and we get *yoga* and *viyoga* respectively. Now, by *saṅkramaṇa* we can get the unknowns.

Bhāskara does not explain the *sūtra* but only gives the method to solve this multiple equations. Kṛṣṇa however gives the solution in detail (*BP.* pp. 243-45). The question is to find two numbers x and y such that

$$x - y + k = u^2 \qquad ①$$

$$x + y + k = v^2 \qquad ②$$

$$x^2 - y^2 + k' = s^2 \qquad ③$$

$$x^2 + y^2 + k'' = t^2 \qquad ④$$

$$\frac{xy}{2} + y = p^3 \qquad ⑤$$

Here k is the *kṣepa* for $(x - y)$ and $(x + y)$, k' is the *kṣepa* for *vargāntara* $(x^2 - y^2)$ and k'' for *vargayoga* $(x^2 + y^2)$.

From ① we have

$$x - y = u^2 - k$$

From ② we have

$$x + y = v^2 - k$$

Multiplying ① and ②

$$\begin{aligned}
u^2v^2 &= (x + y + k)\ (x - y + k) \\
&= (x + y)\ (x - y + k) + k(x - y + k) \\
&= (x + y)\ (x - y) + k(x + y) + k(x - y) + k^2 \\
u^2v^2 &= (x + y)\ (x - y) + k(u^2 - k) + k(v^2 - k) + k^2 \\
&= (x + y)\ (x - y) + ku^2 - k^2 + kv^2 - k^2 + k^2
\end{aligned}$$

$$= (x+y)(x-y) + k(u^2+v^2) - k^2$$

Now $\quad (uv-k)^2 = u^2v^2 + k^2 - 2uvk$

Substituting for u^2v^2, we have

$$\begin{aligned}(uv-k)^2 &= (x+y)(x-y) + k(u^2+v^2) - k^2 + k^2 - 2uvk \\ &= (x+y)(x-y) + k(u^2+v^2-2uv) \\ &= (x+y)(x-y) + k(v-u)^2 \\ &= x^2 - y^2 + k'\end{aligned}$$

If

$$\begin{aligned} k' &= k(v-u)^2 \\ \frac{k'}{k} &= (v-u)^2 \\ \sqrt{\frac{k'}{k}} &= v-u \end{aligned}$$

or

$$v = u + \sqrt{\frac{k'}{k}}$$

in that k' is *vargāntara kṣepa*.

i) अतो वियोगमूलमनेन युक्तं यद्योगमूलं स्यात् । (p.245)– Thus we get $v = u + \sqrt{\frac{k'}{k}}$.

ii) अत: सुष्ठूक्तं ''योगान्तरक्षेपकभाजिताद्यद्वर्गान्तरक्षेपकत: पदं स्यात्'' । – Therefore, it is possible to get the value of 'v' after assuming 'u'.

iii) एवं सिद्धाभ्यां योगवियोगमूलाभ्यां विलोमविधिना योगवियोगौ साध्यौ । – After getting the values of 'u' and 'v', by reverse process, $(x-y)$ and $(x+y)$ can be easily obtained. $u^2 - k = (x-y)$; $v^2 - k = (x+y)$.

iv) तत्र योग: सक्षेपोऽस्य मूलं योगमूलं भवतीति योगमूलं वर्गितं क्षेपोनं स योग: स्यात् । एवं वियोगमूलाद्वियोगोऽपि स्यात् । अथ उक्तं – तयोऽस्तु वर्गौ स्वक्षेपकोनौ हि वियोगयोगाविति । – Sum of the two numbers is equal to square of *yogamūla* less *kṣepa* and difference of two numbers is equal to square of *viyogamūla* less *kṣepa* . So,

$$x + y = v^2 - k \quad , \quad x - y = u^2 - k$$

By *saṅkramaṇa* we have

$$x = \frac{v^2 + u^2 - 2k}{2} \quad , \quad y = \frac{v^2 - u^2}{2}$$

After giving the proof, Kṛṣṇa adds some valid remarks anticipating that some may think that the first three equations would suffice for solving the equation. He says : i) एतयोः राश्योर्मूलत्रयानुरोधेन सिद्धावादवश्यं मूलत्रयलाभः । अवशिष्टपदद्वयलाभे तु न नियमोऽस्ति । – Generally the first three equations are necessary to get the values for x, y and k and there is no such conditions for the latter two ; ii) प्रकृते मूलत्रयानुरोधेन सिद्धयोरव्यक्तराश्योर्याह विधिना पदलाभोऽस्ति तादृशविधेरेव उद्दिष्टत्वात् । – But the other equations should satisfy the assumptions made in the first three equations. iii) तदेव ''सरूपकमव्यक्तमरूपकं'' इत्यादिना सिद्धयोरव्यक्तराश्योर्वियोग-मूलयोगमूलवर्गान्तरमूलान्येव नियतानि न तु पदपंचकमपि नियतमिति सिद्धम् । – Therefore while all the five equations are necessary to solve for x and y, the first three are fixed and the last two can be modified without violating the assumptions made.

The example given is (*BG.* v. 189) :

राश्योर्योगवियोगकौ त्रिसहितौ वर्गौ भवेतां तयो-
वर्गैक्यं चतुरूनितं रवियुतं वर्गान्तरं स्यात्कृतिः ।
साल्पं घातदलं घनः पदयुतिस्तेषां द्वियुक्ताकृति-
स्तौ राशी वद कोमलामलमते षट्सप्त हित्वापरौ ॥

– There are two numbers such that their sum or difference increased by 3 are perfect squares. The sum of their squares decreased by 4 is a square. The difference of their squares increased by 12 is a square. If half their product is increased by the smaller number we get a cube. Again the sum of the 5 roots increased by 2 is a square. Tell me the two numbers other than 6 and 7.

The above problem can be formulated as : Find x and y (other than 6 and 7) such that

$$x - y + 3 = u^2 \quad \text{(a square)} \quad ①$$

$$x + y + 3 = v^2 \quad \text{(another square)} \quad ②$$

$$x^2 - y^2 + 12 = s^2 \quad \text{(another square)} \quad ③$$

$$x^2 + y^2 - 4 = t^2 \quad \text{(another square)} \quad ④$$

$$\frac{xy}{2} + y = p^3 \quad \text{(a cube)} \quad ⑤$$

and finally

$$u + v + s + t + p + 2 = q^2 \quad \text{another square} \quad ⑥$$

6.5.1. Method given by Bhāskara :

$$x - y + 3 = u^2 \Rightarrow x - y = u^2 - 3$$

Similarly $x + y = v^2 - 3$

k is the *kṣepa* for $(x - y)$ and $(x + y) = 3$

$$\textit{vargāntarakṣepa} = k' = 12$$

$$v = u + \sqrt{\frac{k'}{k}}$$

Assume $u = w - 1$

$$v = w - 1 + \sqrt{\frac{12}{3}} = w - 1 + \sqrt{4} = w + 1$$

$$x - y = u^2 - 3 = (w-1)^2 - 3 = w^2 - 2w - 2$$

$$x + y = v^2 - 3 = (w+1)^2 - 3 = w^2 + 2w - 2$$

By *saṅkramaṇa*, we have the bigger number

$$x = \frac{w^2 - 2w - 2 + w^2 + 2w - 2}{2} = w^2 - 2$$

Smaller number

$$y = \frac{w^2 + 2w - 2 - (w^2 - 2w - 2)}{2} = 2w$$

l.h.s. of ① $x - y + 3 = w^2 - 2 - 2w + 3 = w^2 - 2w + 1 = (w-1)^2$, a square

l.h.s. of ② $x + y + 3 = w^2 - 2 + 2w + 3 = w^2 + 2w + 1 = (w+1)^2$, a square

l.h.s. of ③ $x^2 - y^2 + 12 = (w^2-2)^2 - (2w)^2 + 12 = w^4 - 4w^2 + 4 - 4w^2 + 12$
$= w^4 - 8w^2 + 16 = (w^2-4)^2$, a square

l.h.s. of ④ $x^2 + y^2 - 4 = (w^2-2)^2 + (2w)^2 - 4 = w^4 - 4w^2 + 4 + 4w^2 - 4$
$= w^4 = (w^2)^2$, a square

l.h.s. of ⑤ $\dfrac{xy}{2} + y = \dfrac{(w^2-2)(2w)}{2} + 2w = w^3 - 2w + 2w = w^3$, a cube

l.h.s. of ⑥ $u + v + s + t + p + 2 = (w-1) + (w+1) + (w^2-4) + w^2 + w + 2$
$= (2w^2 + 3w - 4) + 2$

According to the last condition

$$2w^2 + 3w - 2 = q^2$$
$$2w^2 + 3w = q^2 + 2$$

Multiplying by 8 and adding 9 we have

$$8(2w^2 + 3w) + 9 = 8(q^2 + 2) + 9$$
$$16w^2 + 24w + 9 = 8q^2 + 25$$
$$(4w+3)^2 = 8q^2 + 25$$

Let
$$4w + 3 = z$$
$$z^2 = 8q^2 + 25$$

Solving the above *varga-prakṛti* equation, we have $z = 15$, $q = 5$ as the least solution set.

$$4w + 3 = z = 15$$
$$4w = 15 - 3 = 12$$
$$w = \frac{12}{4} = 3$$

Thus the two numbers are $w^2 - 2,\ 2w$ i.e. 7, 6.

Since the example says षट्सप्त हित्वापरौ . . . । i.e. "excepting solutions 7, 6", *varga-prakṛti* should be solved again. Auxiliary equation is

$$8(1)^2 + 1 = 3^2$$

$$8(5)^2 + 25 = 15^2$$

		kaniṣṭha	*jyeṣṭha*	*kṣepa*
By *samāsa-bhāvanā*	8	1	3	1
		5	15	25

kaniṣṭha $q = 1 \times 15 + 3 \times 5 = 30$

jyeṣṭha $z = 8 \times 1 \times 5 + 3 \times 15 = 85$

kṣepa $k = 1 \times 25 = 25$

So $8(30)^2 + 25 = (85)^2$

Hence, $4w + 3 = 85$, $w = \frac{82}{4}$ not an integer.

Since w is not an integer, the *varga-prakṛti* is solved again

	kaniṣṭha	*jyeṣṭha*	*kṣepa*
8	1	3	1
	30	85	25

kaniṣṭha $q = 1 \times 85 + 3 \times 30 = 175$

jyeṣṭha $z = 8 \times 1 \times 30 + 3 \times 85 = 495$

Now $8(175)^2 + 25 = (495)^2$

$4w + 3 = 495$

$$w = \frac{495 - 3}{4} = 123.$$

Thus the two numbers x and y are $w^2 - 2, 2w$ i.e.

$123^2 - 2, 2 \times 123$ i.e. 15127, 246.

6.6. Vargakuṭṭaka

The last few examples in the *Madhyamāharaṇa* chapter deal with *vargakuṭṭaka* and *ghanakuṭṭaka.*[19] In this section *vargakuṭṭaka* alone is taken up for discussion.

The indeterminate equation of the type $x^2 = by + c$

is called *vargakuṭṭaka.* Re-writting $\frac{x^2 - c}{b} = y$

This closely resembles the linear *kuṭṭaka* equation in as much as the problem reduces to finding a square which when reduced by c will be exactly divisible by b.

Earlier authors like Brạhmagupta (*Br.Sp.* XVIII 79) have adopted the method of assuming suitable arbitary values of y and then solving for 'x'. Though Pṛthūdakasvāmin, the commentator of *Br.Sp.* presupposes the existence of infinite solutions, neither Brahmagupta nor he have given any general solution.[20]

Bhāskara was the first to give a general solution as well as the rule for solving the *vargakuṭṭaka* (*BG.* v. 196) :

वर्गादियों हरस्तेन गुणितं यदि जायते ।
अव्यक्तं तत्र तन्मानमभिन्नं स्याद्यथा तथा ।
कल्प्योऽन्यवर्णवर्गादिस्तुल्यं शेषं यथोक्तवत् ॥

– When we multiply by the divisor of the square we should assume square of some unknown and the remaining process should be given as before so that the value of x will be integral.

19. *Ghanakuṭṭaka* is omitted, since Kṛṣṇa does not add anything new to what has been already said by Bhāskara.

20. Datta and Singh, *op.cit.*, Vol. II, p. 252

Bhāskara (*BG.* vv. 198-200) gives the method of earlier authors in the *sūtras*. Kṛṣṇa elaborately explains the method contained in the above verses. Initially Kṛṣṇa defines (*BP.* p. 250) *vargakuṭṭaka* as : इह वर्गकुट्टके को वर्ग: उद्दिष्टक्षेपेण युतो ऊनो वा उद्दिष्टहरभक्त: शुध्यतीति आलापो अस्ति । – In the *vargakuṭṭaka* the square of a number with a positive or negative *kṣepa* is divisible by the divisor without remainder. Let the equation be $x^2 = by + c$. (*BP.* p. 252) : द्वितीयपक्षे हरतुल्यो वर्णाङ्क: क्षेपतुल्यानि रूपाणि धनमृणं वा भवतीति सिद्धम् । पूर्वपक्षस्य वर्गात्मकत्वात् पदे गृहीते द्वितीयपक्षेऽपि पूर्वपक्षसमत्वात् वर्ग एवेति कस्यचिदन्यवर्णस्य वर्गेण समं कर्तुं युज्यते । – c can be positive or negative. Since the left side is a square, the right side should also be a square. Obviously $(by + c)$ should be a square so that x is an integer.

$$x^2 = by + c$$
$$x = \pm \sqrt{by + c}$$

<u>**Case 1**</u> (*BG.* v. 198) : हरभक्ता यस्य कृति: शुध्यति सोऽपि द्विरूपपदगुणित: ।
तेनाऽऽहतोऽन्यवर्णो रूपपदेनान्वित: कल्प्य: ।।

– If c is a square, then y should be assumed as $bz^2 + 2\sqrt{c}\,.z$ so that $by + c$ is a square.

$$x^2 = b(bz^2 + 2\sqrt{c}\,.z) + c$$
$$= b^2z^2 + 2bz\sqrt{c} + c$$
$$= (bz \pm \sqrt{c}\,)^2$$

(*BP.* p. 252) : ननु रूपयुते रूपोने वा अन्यवर्णो कल्पिते तस्य वर्गे क्रियमाणे अन्यवर्ण-वर्गोऽन्यवर्णो रूपाणि चेति खण्डत्रयं स्यात् । तत्र समशोधनेन रूपनाशे खण्डद्वयमवशिष्यते । ... इदं खण्डद्वयमपि यथा हरभक्तं शुद्ध्यति तथाङ्क: कल्प्य: । अत एवोक्तं यस्य कृतिर्हरभक्तशुद्ध्यति अपि च सोङ्को द्विरूपपदगुणितोऽपि शुद्ध्यति तदा तेनाङ्केनाहतोऽन्यवर्ण: कल्प्य इति । – By expanding the square, we have three terms. Cancelling c on both sides only two terms remain. Obviously the right hand side is divisible by b. Since

$$x^2 = by + c$$
$$by + c = b^2z^2 + 2bz\sqrt{c} + c$$
$$by = b^2z^2 \pm 2bz\sqrt{c}$$

Thus y becomes an integer since all the terms on the r.h.s. are divisible by b. Thus it is said that if c is a square, assume y to be equal to $bz^2 \pm 2\sqrt{c}\,.z$

Case 2 : अथ यदि द्वितीयपक्षरूपाणां पदं न लभ्यते तदा तृतीयपक्षो हि मूलदः कल्पनीयः यतोऽस्य पदेन प्रथमपक्ष साम्यं विधेयमस्ति । – When c is not a square, assume x to be $\sqrt{by+c}$. Again $by+c$ should be a square for x to be an integer. Assume

$$by + c = (bm + n)^2 = b^2m^2 + 2bmn + n^2$$

Obviously the first two terms are divisible by b. For the third term we can argue thus :

$$by + c = b^2m^2 + 2bmn + n^2$$
$$by = b^2m^2 + 2bmn + n^2 - c \qquad ①$$

(*BP.* p. 253) : अतस्तृतीयपक्षे रूपवर्गस्तथा कल्प्या यथा तस्य द्वितीयपक्षरूपैः सहान्तरमेकादि-गुणितहरतुल्यं स्यात् । यतः तथा सति तत् शेषं हरभक्तं शुध्येदेवेति द्वितीयवर्ण-नमानमभिन्नं स्यात् । – In equation ①, the term $(n^2 - c)$ alone is not divisible by b. Therefore it should be made divisible by b, by assuming a new variable k. Thus y becomes an integer since all the terms on the right hand side (r.h.s.) are divisible by b. Let

$$\frac{n^2 - c}{b} = k \Rightarrow \quad n^2 = bk + c$$

Let k be any number such that $bk + c = n^2$, then in ①

$$by = b^2m^2 + 2bmn + bk + c - c$$
$$= b^2m^2 + 2bmn + bk$$

which is divisible by b. Therefore Kṛṣṇa says : यदि रूपाणि हरतष्टानि मूलदानि स्युस्तदा तत्पदेनान्वितोऽन्यवर्णः कल्प्य इति । – to find k such that $bk + c = n^2$; then $by + c$ should be equated to $(bm + n)^2$ where m is any convenient number satisfying the equation $(by + c) = (bm + n)^2$

Case 3 : When the equation is of the form $ay^2 = bx \pm c$ multiplying by a ,

$$a^2y^2 = abx \pm ac$$

putting $ay = u,\ \ ax = v,$

we have $u^2 = bv \pm ac$

This is the same as the *vargakuṭṭaka* described above.

Remarks : (i) Kṛṣṇa adds a new idea to the much discussed words '*hatvā kṣiptvā*'[21]. (*BP.* p. 255) – अत्र क्षिप्त्वेति यदुक्तं तत्प्रथमराशौ सरूपे कल्पिते सति द्रष्टव्यम् । यद्वा पक्षौ तदिष्टेन निहत्य किंचित् क्षेप्यं तयोरित्येतदर्थकस्याद्यसूत्रस्य ñ_maHS§hËdm{j]ËdqV E – According to him '*hatvā*' refers to multiplication – in the above example, ay^2 is multiplied by a to make it a square. And '*kṣiptvā*' refers to some *kṣepa* added to a multiple of the divisor as mentioned in case 1 and case 2.

(ii) Bhāskara has also added the phrase आलापित एव हर: । – "the divisor should remain as it has been stated ". In the *vargakuṭṭaka* equation given above, even after multiplying by a throughout and putting $ay = u$ and $ax = v$, the equation still remains as $u^2 = bv \pm ac$ i.e. the divisor b of the initial equation $ay^2 = bx + c$ is retained in this equation also as the divisor. Now the question may be asked as to why the assumption be such that the divisor remains the same. Kṛṣṇa says (*BP.* p. 254) : आलापित एव हर इति यदुक्तं तल्लाघवार्थं द्रष्टव्यम् । – "for easy working". He also adds that : आलापितहरेपि पक्षसाम्यं न हीयते । – by doing so, the equation is not affected.

An example given by Bhāskara (*BG.* v. 202) makes the above assumption clear : Find y where $5y^2 + 3$ is divisible by 16 without remainder

$$5y^2 + 3 = 16x \Rightarrow 5y^2 = 16x - 3$$

21. For a discussion on this, refer Datta and Singh, *op.cit.*, Vol. II, p. 255 fn. and H.T. Colebrooke, *op.cit.*, p. 263 fn.

Multplying by 5 throughout,

$$25y^2 = 80x - 15 \qquad ①$$

Let
$$5y = u \; ; \; 5x = v$$
$$u^2 = 16v - 15$$

Following the *sūtra* (*BG.* v. 200) :

हत्वा क्षिप्त्वा च पदं यत्राऽऽद्यस्येह भवति तत्रापि ।
आलापित एव हरो रूपाणि तु शोधनानि सिद्धानि ॥

– The divisor is retained as 16 as before. Using case 2 and solving as above, $v = 4w^2 + w + 1$, $u = (8w + 1)$

$$u^2 = 16v - 15$$

So,
$$5y = 8w + 1$$

By *kuṭṭaka* we have $y = 8t + 5$, $w = 5t + 3$, $t = 0, \pm 1, \pm 2 \ldots$ On the other hand we can also deal with equation ① in the following manner

$$5y = \sqrt{80x - 15}$$
$$25y^2 = 80x - 15$$

Let $80x - 15$ be $= (40m + 15)^2$

$$80x - 15 = 1600m^2 + 1200m + 225$$
$$80x = 1600m^2 + 1200m + 240$$
$$x = 20m^2 + 15m + 3$$

Since we obtain x as an integer,

$$25y^2 = (40m + 15)^2$$
$$5y = 40m + 15$$
$$y = 8m + 3$$

When $m = 0$; $y = 3$; $m = 1$; $y = 11$; . . .

Therefore other assumptions can also be made. According to Kṛṣṇa, Bhāskara has retained the divisor for easy working.

6.7. *Bhāvita*:

Bhāvita as a mathematical term refers not only to the product of two unknown variables such as xy, ab and so on but also to the genre of equations dealing with products. The equations of *bhāvita* type have been defined by Bhāskara as (*BG.* p. 44) : यत्र भावितस्य तद्भावितमिति बीजचतुष्टयं । – type (fourth) of equations involving product of two or more unknowns. Kṛṣṇa defines it as (*BP.* p. 156) : यत्र तु भावितमधिकृत्य साम्यं क्रियते तद्भावितमित्युच्यते – where the equations deals with product of unknowns it is called *bhāvita.*

The earliest mention of such an equation $xy = 3x + 4y \pm 1$, is found in the *BM.* (VI.4 (27#^{r})) In his *Br.Sp.* (XVIII. 62-3), Brahmagupta also quotes an unknown author's example – $axy = bx + cy + d$. Though Mahāvīra has not treated equations of this type, his *G.S.S.* (VI. 35, 284) has two problems which are similar to the equations mentioned above. This type of equation has been discussed by Śrīpati also in his treatise.

Tracing the history of geometrical representations in Algebra, T.A. Saraswati Amma in her book on Geometry writes : "The practice of representing and solving algebraic and arithmetical problems geometrically is as old as geometry itself . . . The *Śulba sūtra* writers were primarily interested in geometrical constructions, the implied algebraical truths coming in by the side door. But with the later mathematicians the algebraical results are the most important, the geometrical figures being merely an aid to make the algebraical results the more convincing, or to prove the results."[22] The

22. *Geometry in Ancient and Medieval India*, Motilal Banarasidass, Delhi, 1979, pp.220-21.

interesting feature about this chapter is that the proof is given both algebraically and geometrically. In Brahmagupta, Mahāvīra and Bhāskara such use of gemoetry is limited. Later, commentators on Bhāskara do give diagrammatical corroborations of algebraical formulae.

Bhāskara's aim was to give integral solutions. He gives two rules, the first of which follows the method of Brahmagupta.

The example given by Bhāskara is (*BG.* v. 204) :

चतुस्त्रिगुणयो राश्योः संयुतिर्द्वियुता तयोः ।
राशिघातेन तुल्या स्यात्तौ राशी वेत्सि चेद्वद ॥

– Solve for (x, y) : $4x + 3y + 2 = xy$

Bhāskara himself states that such equations can be solved both algebraically and geometrically. (*BG.* p. 125) : अस्योपपत्तिः । सा च द्विधा सर्वत्र स्यादेका क्षेत्रगताऽन्या राशिगतेति । However Kṛṣṇa specifically makes use of diagrams to solve the equation.

Kṛṣṇa's solution (*BP.* pp. 258-62) : Before providing his proof, Kṛṣṇa makes a clear statement : अत्रोपपत्तिः आचार्यैः लिखितोऽस्ति । किं तु लेखकादि-दोषादुपदेशविच्छित्त्या च संप्रति सा न स्वकार्यक्षमा । अत इयं भावितोपपत्तिः विविच्योच्यते । – Here the rationale has been written by the Ācārya. But due to the errors of the scribes and a break in the tradition of instruction, that (rationale) is not now capable of fulfilling its purpose. Hence the rationale for *bhāvita* is given in detail .[23]

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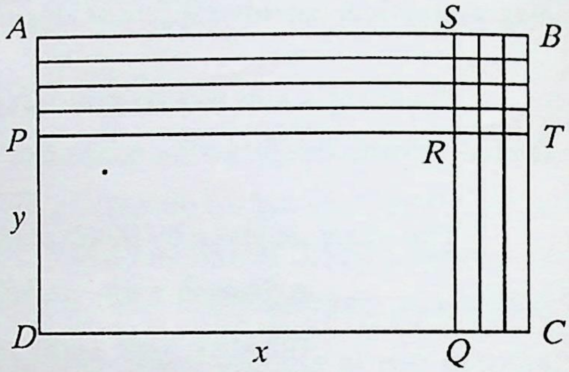
23. When explaining the example given by Bhāskara, Mm. Sudhakara Dvivedi recommends that the *Bījāṅkūra* of Kṛṣṇa (*BG.* p. 126), should be referred to since there were some errors in the original : अत्र मूले लेखकाध्यापकाध्येतृदोषैः काचित् त्रुटिरस्ति तदर्थं कृष्णदैवज्ञकृता नवाङ्कुराख्या बीजगणितटीका विलोक्या ।

Commenting on this, Mm. Muralidhara Jha in his expository notes says that there is not much contribution by Kṛṣṇa in this regard except explaining in detail: ''रूपचतुष्टयोनकालके स्वाङ्कगुणे'' वा ''कालके रूपचतुष्टयोनेऽथ स्वाङ्कगुणे'' इह न काचित् त्रुटिरस्ति । वस्तुतो नवाङ्कुरटीकाकारस्य कृष्णदैवज्ञस्य वाक्यबाहुल्यतोऽन्यत् किमपि न सारमिति विज्ञैर्विवेचनीयम् ।

Taking the same equation $xy = 4x + 3y + 2$, भावितं च समकर्णायत चतुर्भुजक्षेत्रफलं । तत्र वर्णौ भुजकोटी । – The product $(x\ y)$ evidently denotes the area of a rectangle with sides x (*bhuja*) and y (*koṭi*).

The area of rectangle *ABCD* in the adjoining figure is comprised of rectangle *ASRP* + rectangle *SBTR* + rectangle *PRQD* + rectangle *RTCQ*.



Rectangle *ABTP* (from figure) = $4x$

Rectangle $SBCQ = 3y$

(*BP*. p. 259) : उभयथाऽतिवर्णाङ्का इतितुल्यैरूपैरूनं यावत्तावत् चतुष्टयं कालकत्रयं च क्षेत्रमध्ये प्रदर्शितं भवति । – When the two rectangles are added, the rectangle *SBTR* has been counted twice and so one has to be discarded. From the figure we have $xy = 4x + 3y - 12$ + rectangle *PRQD*

But it is given that $xy = 4x + 3y + 2$

i.e. $4x + 3y - 12$ + rectangle $PRQD = 4x + 3y + 2$

i.e. Rectangle $PRQD - 12 = 2$

i.e. Rectangle $PRQD = 12 + 2 = 14$

(*BP*. p. 260) : ते च तस्य लघुक्षेत्रस्य फलम् । तद्भुजयोर्वधाज्जातम् । अत इष्टमेकं भुजं प्रकल्प्य तेन क्षेत्रफले भक्ते यल्लभ्ये तद्वितीयो भुजः स्यात् । – The area of the smaller rectangle *PRQD* (= 14) is the product of the two sides $(p \times q)$. The area divided by one of the sides will give the other side.

$$\text{Area} = bhuja \times koṭi = p \times q$$

Assume that one side

$$p = 1 , \quad \text{then} \quad q = \frac{14}{1} = 14$$

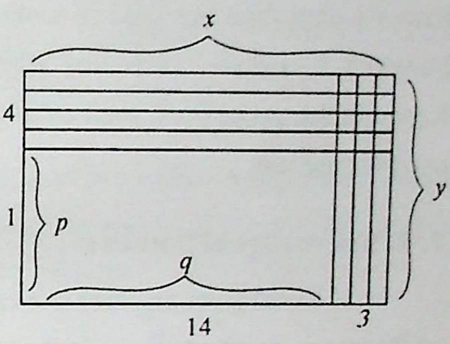
$$p = 2 , \quad \text{then} \quad q = \frac{14}{2} = 7$$

If we take $p = 1, \quad q = 14$

then $y = 5, \quad x = 17$

If we take $p = 2, \quad q = 7$

then $y = 6, \quad x = 10$



Kṛṣṇa concludes : अत उपपन्नमिष्टफलाभ्यां स्वेच्छया संयुतौ वर्णाङ्कौ व्यत्ययाद्वर्णयोर्मानि ज्ञातव्ये इति । – Thus the equation can be solved using the above geometrical figures.

Many interesting concepts to solve different illustrations of linear and quadratic equations with many variables have been dealt with in the foregoing pages. The much discussed verse – *śadaṣṭaśatakāḥ . . .* , has been analysed. The assumption made by Bhāskara not being sufficient, Kṛṣṇa has made a few more assumptions, to make the problem valid. In this section Kṛṣṇa has added two simple rules given by his *guru* Viṣṇu Daivajña to solve a couple of problems. It is also the first time, that Kṛṣṇa has introduced two examples of his own. In the *Madhyamāharaṇa* section, quadratic equations (or equations with higher powers of x) have been cleverly reduced to the form of *varga-prakṛti* in order to solve them. In *Bhāvita* section, Bhāskara has explained *bhāvita* equations with diagrams which Kṛṣṇa has elucidated.

CHAPTER - 7
KṚṢṆA'S ERUDITION : AN APPRAISAL

In the foregoing chapters, Kṛṣṇa's *BP* as a dependable commentary on *BG* has been discussed elaborately. A careful study of the text reveals Kṛṣṇa's expertise not only in mathematics but also in the other fields of Sanskrit literature, including philosophy. Also, innovations made by Kṛṣṇa in the process of mathematical discussions and some shortcomings are pointed out.

7. 1. Knowledge of Darśanas :

a) Kṛṣṇa gives an elaborate explanation for the first benedictory verse of Bhāskara in which he finds the core of Sāṅkhya and Yoga philosophy :

उत्पादकं यत्प्रवदन्ति बुद्धेरधिष्ठितं सत्पुरुषेण साङ्ख्याः ।
व्यक्तस्य कृत्स्नस्य तदेकबीजमव्यक्तमीशं गणितं च वन्दे ।।

"What the learned calculators (*sāṅkhyāḥ*) describe as the originator of intelligence, being directed by a wise being (*sat puruṣa*) and which alone is the primal cause (*bīja*) of all knowns (*vyakta*), I venerate that invisible God as well as that science of calculation with unknowns."[1]

Here Bhāskara has used *śleṣa* or double entendre very cleverly. Using the word '*sāṅkhya*', Kṛṣṇa explains, that the Ācārya has brought forth the essence of the philosophy related to Īśvara on the one hand and the crux of algebra which is known as *avyakta gaṇita* on the other.

On the *īśapakṣa,* the verse would mean that Īśvara is the ruling power and the sages know Him to be the cause of knowledge and He is the One element of all which is apparent. On the *gaṇitapakṣa,* it would mean unapparent computation which is the means of understanding ; it is the single element of all which is apparent.

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1. Datta and Singh, *op.cit.,* Vol. II. p.2.

Here Kṛṣṇa clarifies that the word *sāṅkhya*, stands for *seśvara sāṅkhya* of Patañjali and not *sāṅkhya* of Kapila, since the latter school does not accept the existence of God ; according to them all creation originates from the *pradhāna* (*prakṛti* – primal cause). In support of this interpretation Kṛṣṇa quotes from Īśvara Kṛṣṇa's *Sāṅkhyakārikā* (v. 57) :

वत्सविवृद्धिनिमित्तं क्षीरस्य यथा प्रवृत्तिरज्ञस्य ।
पुरुषविमोक्षनिमित्तं तथा प्रवृत्तिः प्रधानस्य ।।

b) Elaborating on this Kṛṣṇa interprets the word *sāṅkhyāḥ* as realised Souls through *sādhanacatuṣṭaya* starting with *viveka* (*BP.* p. 4) – साङ्ख्या आत्मज्ञानिनः ... विवेकादि साधनचतुष्टय संपत्तिमता ।

c) He also explains the *pariṇāmavāda* (theory of transformation) of Sāṅkhya philosophy and the *vivartavāda* (theory of appearance) of Advaita in this context.

d) These *Taittirīya Upaniṣad* statements : यतो वा इमानि भूतानि जायन्ते । (III.1.1.), तत्सृष्ट्वा तदेवानुप्राविशत् । (II.6.1.) and तस्माद्वा एतस्मादात्मन आकाशः संभूतः । (II.1.1.) are cited by Kṛṣṇa while explaining that from the unmanifest seed (*bīja*) everything get manifested (*vyakta*) (*BP.* p. 4) : समस्तस्य व्यक्तस्य कार्यजातस्य एकमसाधारणं बीजमुपादानमित्यर्थः ।

From the above references, it is evident that Kṛṣṇa had deep knowledge in many schools of Indian philosophy.

7.2. Knowledge of Vyākaraṇa :

Kṛṣṇa was well versed in Vyākaraṇa also. Wherever necessary, he breaks the long compound words to make it intelligible for the pupils. To quote a few :

a) while explaining the addition of zero as $a + 0 = a$; $0 + a = a$, and subtraction of zero $a - 0 = a$; $0 - a = -a$: Kṛṣṇa says (*BP.* p. 24) :

खेन युक्तं खयुक्तं, खे युक्तं खयुक्तमित्युदाहरणद्वयमपि द्रष्टव्यम् । एवं खच्युतमित्यत्रापि तृतीयपञ्चमीतत्पुरुषाभ्यामुदाहरणद्वयं द्रष्टव्यम् । – Applying the rule of *tatpuruṣasamāsa,* the term *khayuktam* should be understood as 'added by zero' and 'added to zero'. In the same way the term *khacyutam* should be understood as 'subtracted by zero' and 'subtracted from zero'.

b) Defending the usage of ताः लब्धयः in *BG.* v. 29, Kṛṣṇa (*BP.* p. 42) explains : ताः लब्धय इत्यत्र तच्छब्दस्य विधीयमानलिङ्गता " शैत्यं हि यत्सा प्रकृतिर्जलस्य " इत्यादौ प्रसिद्धा । " दैवे युगसहस्रे द्वे ब्राह्मः कल्पौ तु तौ नृणाम् " इत्यस्य व्याख्यावसरे लिखितं च क्षीरस्वामिना - सर्वनाम्ना विधीयमानानूद्यमानलिङ्गग्रहणे कामचारः इति । – When *sarvanāma śabdas* are used they can take up any gender, is a rule given by Kṣīrasvāmī in his commentary on *Amarakośa* (I.3.21). This justifies the use of *sarvanāma śabda* – '*tāḥ*' in feminine gender along with the word '*labdhayaḥ*' in masculine. In this context, Kṛṣṇa's intention is as folows: In the phrase, यैः वर्णैः ... यैः रूपैः ... लब्धयस्ताः ... the relative pronoun *tad* may take the gender of the antecedent (*varṇa* and *rūpa*) or of the predicate (*labdhi*, feminine).

c) In *Kuṭṭakādhyāya*, while discussing the formation of *vallī* (*BG.* v. 58), Kṛṣṇa justifies the usage of the word *phalāni* as (*BP.* p. 88) : . . . परस्पर भजनेष्वागतानि फलानि अधोऽधो निवेश्यानि । – dividing again and again, the resultant quotients should be placed one below the other. Here he explains that quotient can be one, two or many. Hence the use of the word *phalāni* : फलं च फले च फलानि च फलानि । द्वन्द्वैकशेषः ।[2]

d) Explaining the word '*taṣṭaḥ*' (तष्टः) in the *sūtra* (*BG.* v. 59), Kṛṣṇa says that it means the action of reduction (*BP.* p. 88) : तक्षत्वक्षतनूकरणे कर्मणि क्तः । that is by the rule of Pāṇini (III. 4.17) : तयोरेव कृत्यक्तखलर्थाः । and by the *sūtra* (III.2.102) : निष्ठा, the word '*takṣatvakṣa*' changes to '*taṣṭaḥ*'.

2. This is according to the rule of Pāṇini : चार्थे द्वन्द्वः (II.2.29) and सरूपाणामेकशेष एकविभक्तौ । (I.2.64).

e) While explaining the words *kadambam* and *śilīndhram* occurring in *BG.* v. 110, Kṛṣṇa says (*BP.* p. 164) : कदम्बस्य पुष्पं कदम्बम् । अवयवे च प्राण्योषधिवृक्षेभ्य इत्यण् । पुष्पमूलेषु बहुलमिति तस्य लुक् । शिलीन्ध्रायाः पुष्पं शिलीन्ध्रम् । लुक्तद्धितलुकीति स्त्रीप्रत्ययलोपः । ... मालत्या पुष्पं मालती । मल्लिकायाः पुष्पं मल्लिकेतिवन्न स्त्रीप्रत्ययलोपः ।

According to this, the word *kadambam* is derived from the rule of Pāṇini (IV. 3.166) : लुपश्च ; explaining this, *Vārttika* adds (2950) : पुष्पमूलेषु बहुलम् । The word *śilīndhram* is derived under the rule (I. 2.49) : लुक्तद्धितलुकि । Hence, the *strīpratyaya* gets dropped. It may be noted that the words *mālatī* and *mallikā* retain their *strīpratyaya.*

f) Commenting on verse 111 of *BG,* Kṛṣṇa quotes Vijñāneśvara the commentator on *Yājñavalkyasmṛti* who in the *vyavahāra kaṇḍa* states that the word '*pañca*' meaning number 5 takes '*kapratyaya*' and changes into '*pañcaka*'[3] (*BP.* p. 165) : प्रतिमासं पञ्चवृद्धिर्यस्येति पञ्चकमिति विज्ञानेश्वरेण व्यवहाराध्याये विवृतं संज्ञायां कप्रत्यय विधानात् ।

g) Explaining the word *śrutipathāt* in *BG* v. 128, Kṛṣṇa applying Kātyāyana's *Vārttika* (1474) ल्यब्लोपे कर्मण्यधिकरणे च । under Pāṇini's rule भुवः प्रभवः, states that the word *śrutipathāt* means 'taking the diagonal path' (*BP.* p. 183) : श्रुतिपथादिति ल्यब्लोपे पञ्चमी । श्रुतिपथमाश्रित्येति तदर्थः ।

h) Kṛṣṇa (*BP.* p. 225) on the word *ṣaḍaṣṭaśatakāḥ* in (*BG.* v. 168) : षट् अष्टौ शतं च धनं विद्यते येषां ते षडष्टशताः अर्श आदिभ्योऽच् इति मत्वर्थीयोऽच् प्रत्ययः । त एव षडष्टशतका इत्यत्र स्वार्थे कप्प्रत्ययः । – the '*kapratyaya*' is added here to denote that the money is possessed by the people.[4]

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3. Refer Pāṇini, V.3.97 : संज्ञायां क्च ।
4. *ibid.,* V.2.127 : अर्श आदिभ्योऽच् and V.3.97 : संज्ञायां क्च ।

7.3. His familiarity with Prosody :

Kṛṣṇa's knowledge in the field of prosody becomes evident from the fact that he has identified all the metres used by Bhāskara. We find that neither Bhāskara himself nor Sūryadāsa, the other commentator on *BG* has undertaken this task.

a) The metres employed by Bhāskara as recorded by Kṛṣṇa are - *Śālinī*, *Upajātikā*, *Bhujaṅgaprayāta*, *Indravajrā*, *Vasantatilakā*, *Mandākrāntā*, *Gīti*, *Āryā*, *Upagīti*, *Anuṣṭubh*, *Rathoddhatā*, *Śārdūlavikrīḍita*, *Simhoddhatā*, *Vamśastha* and *Mālinī*.

b) Commenting on *BG.* v. 55, Kṛṣṇa identifies it as in *Upagīti* metre. He points out that if the fourth *pāda* were to have the extra word *yatra,* as found in an alternative reading, the metre would be *Udgīti*. (*BP.* p. 83) : अत्र चतुर्थचरणे यत्र कृतौ तत्र किं पदं ब्रूहि इति पाठे सा उद्गीतिर्ज्ञेया ।

c) On *BG.* v. 48, Kṛṣṇa explains that it is in *Gīti* metre. If the reading were to miss the syllable '*ca*' as found in some manuscripts, there will be *chandobhaṅga* (*BP.* p. 76) : अत्र द्वितीयं गीतौ तिथिषु पञ्चानामिति बहवः पठन्ति । तत्र तिथिषु च पञ्चानामिति पठनीयम् । अन्यथा छन्दो भङ्गात् ।

d) In the same context (*BG.* vv. 47-8), Kṛṣṇa notices another instance of *chandobhaṅga* relating to the number of terms in the square of a *karaṇī* thus (*BP.* p. 77) : द्वयोः सरूपैकेति आर्यां कल्पयित्वा सूत्रमध्ये पठन्ति । तदशुद्धम् । करणीति तिसृणां तिस्र इत्यादेरग्रिमग्रन्थस्यानन्वयात् । न ह्येकमेव वाक्यं श्लोकचूर्णिकात्मकमिति रीतिरस्ति । पूर्वार्धे छन्दोभङ्गाच्च । According to Kṛṣṇa both the above mentioned verses are in *Gīti* metre. Kṛṣṇa's point is as follows : some people regard the beginning of the *Vāsanā – karaṇaī varga rāśau . . . dvayoḥ sarūpaika*, as an *Āryā* verse; but it is wrong. Kṛṣṇa argues that this would not be correct in the light of the subsequent *sūtra*, where if the square has six *karaṇī* terms its square root would have three *karaṇī* terms and so on.

e) The following is another instance of *chandobhaṅga* (*BG.* v. 53) : on this Kṛṣṇa points out, that the relevant word should be *traya* and not *tritaya* since it would result in one extra syllable, not fitting the *Āryā* metre. (*BP.* p. 81) : अत्र करणीत्रितयं कृतौ (सखे) यत्रेति केचित् पठन्ति, तदशुद्धम् । मात्राधिकेन छन्दोभङ्गात् ।

7. 4. His knowledge of Laukikanyāyas :

That Kṛṣṇa was quite familiar with *laukikanyāyas* is evident from the various maxims he quotes to support his views :

1) *Kaimutikanyāya* (*BP.* p. 6) : ईशस्य समस्तकार्यजनकत्वं वदता तत्प्रमाणस्य ग्रन्थसमाप्तिप्रचयादि रूपं फलं कैमुतिकन्यायेनैव सूचितम् । . . . ईशस्तु सर्वं कर्तुं शक्तः स्वप्रणतस्य सर्वमिष्टं विदध्यात् । ग्रन्थसमाप्तिप्रचयादि रूपं किमुत इति । – That is, God is capable of doing everything, and capable of fulfilling the wishes of the seeker. Kṛṣṇa applies the *kaimutika* maxim here and says that it is not difficult to complete the text with the blessings of the Lord.

2) *Sūcīkaṭāhanyāya* (*BP.* p. 31) : तथापि करणीषड्विधस्य अतिकठिनतया तन्निरूपणे प्रयासबाहुल्याद् अव्यक्तषड्विधनिरूपणे च प्रयासलाघवात् सूचीकटाहन्यायेन अव्यक्तषड्विधं प्रथमतो निरूपयति । – According to the *needle and kettle maxim* (*sūcikaṭāhanyāya*), the easier job should be done first and the difficult, later. Applying this, Kṛṣṇa says that since the six mathematical operations of the unknown (*avyaktaṣaḍvidha*) is easier, it is taken up for discussion first, followed by discussion on six mathematical operations on *karaṇī* (*karaṇīṣaḍvidha*).

3) *Dehalīdīpanyāya* (*BP.* p. 31) : अत्र स्यादिति पदमुत्तरदलस्थमन्वेति देहलीदीप-न्यायेन पृथक् स्थितिः स्यादिति पाठः । – Like the lamp on the threshold (*dehalīdīpanyāya*), throwing light both inside and outside, here, the word '*syāt*' applies to both the first and second sentences in the particular context.

7.5. *Apapāṭhas* noticed by Kṛṣṇa :

It is evident from the text *BP* that Kṛṣṇa has gone through the original manuscript very carefully and collated with other manuscripts available to him. Often he compares two or more readings and chooses the best, giving valid reasons for the same. Some of his observations are as follows:

1) Kṛṣṇa says (*BP.* p. 81) : अत्र रुद्रा इति पाठे, नागर्तवश्चतुर्गुणा इति न प्रतीयन्ते । अतो रुद्रनागर्तव इति पाठः साधीयान् । – Here, *rudra* is apt in the place of *rudrā,* since the word *caturguṇāḥ* (multiplied by four) is to be applied for all the terms in the compound सूर्यतिथीषुरुद्रनागर्तवः । If *rudrā* is considered then the term *caturguṇāḥ* will not apply to the latter two terms in the compound *viz., nāgartava.*

2) In the verse 55, Kṛṣṇa points out that the reading *aśītirdviśatītulyāḥ* is not correct. He says (p.83) : अशीतिरिति रेफान्तः पाठो न युक्तः । According to the Pāṇinian rule, सुपो धातुप्रातिपदिकयोः। (II. 4.71), the reading *aśīti* is the right one[5], since – एतयोः अवयवस्य सुपः लुक् स्यात् – there is elision (*luk*) of the case-suffix, since it is a compound.

3) In the chapter on *Kuṭṭaka* there is a *pāṭhabheda* that Kṛṣṇa takes note of and explains (*BP.* p. 106) :

अत्रोत्तरार्धे ऋणभाज्योद्भवे तद्वद्भवेतामृणभाजक इति अपि पाठः क्वचित् दृश्यते । अस्यार्थः । योगजे गुणाप्ती स्वतक्षणात् शुद्धे वियोगजे भवतः तद्वदृणभाज्योद्भवे भवतः। तद्वदृणभाजकेऽपि गुणाप्ती भवतः । क्षेपभाज्यहाराणामन्यतमे ऋणे सति पूर्वसिद्धे गुणाप्ती स्वतक्षणात् शोध्ये इत्यर्थः। एवं द्वौ चेदृणगतौ तदा पुनरपि स्वतक्षणात् शोध्ये इत्यर्थः । एवं त्रयाणामप्यृणत्वे त्रिवारं स्वतक्षणात् शोध्ये इत्यर्थः। **अयमपपाठः** । न हि भाजकस्य ऋणत्वे धनत्वे वास्ति कश्चिदङ्कतो विशेषः येनोपायान्तरमारभ्येत धनर्णताव्यत्यासमात्रं लब्धेः । भाज्यस्य तु ऋणत्वे धनत्वे च क्षेपयोगे क्रियमाणे त्वङ्कतोऽपि विशेषः इति तस्यर्णत्वे उपायान्तरमारम्भणीयमेव ।

Here Kṛṣṇa points out that if we take the reading *ṛṇabhājya* in the place of *dhanabhājya* and *ṛṇabhājaka* in the place of *ṛṇabhājyaja* it will

5. In his edition of *BG* (p. 24), Mm. Sudhakara Dvivedi retains the reading *aśītiḥ* keeping the *rephānta.*

lead to wrong solution.[6] Continuing this discussion on *BP.* p.118 Kṛṣṇa shows how this would work out wrongly in the given problem. This has been already discussed in Chapter **3.5.2.**

4) On the next *sūtra* in the same *Kuṭṭakādhyāya*, Kṛṣṇa clarifies thus (*BP.* p. 108) – अत्र पुस्तकेषु गुणलब्ध्योः समं ग्राह्यमित्यादिश्लोकेर्धस्य योगजे तक्षणाच्छुद्धे इत्यतः प्राक् पाठः दृश्यते । स तु लेखकदोष इति प्रतिभाति । पुस्तकपाठक्रमस्वीकारे तु गुणलब्ध्योः समं ग्राह्यमित्यत्र प्रकारान्तरार्थं प्रवृत्तस्य हरतष्टे धनक्षेपे इत्येतस्य सूत्रस्य व्यवधानं स्यात् । उदाहरणक्रमविरोधश्च स्यात् । लीलावतीपुस्तकेषु पुनरस्मल्लिखितक्रम एव अस्ति युक्तश्चायमिति प्रतिभाति ।

Here Kṛṣṇa points out that, if the order of the *sūtras* of *BG.* (vv. 62-4), gets altered in their order, it would not be appropriate, as it would not suit the sequence of examples given by Bhāskara ; he further adds that the sequence of the *sūtras* given by himself, would agree with the text *Līlāvatī.*

5) In the example on *Sthirakuṭṭaka* (*BG.* v. 75), Kṛṣṇa makes this observation (*BP.* p. 123) : अत्राचार्यव्याख्याने युगावमानि भाज्य इत्यत्र कल्पशब्दस्थाने युगेति लिखनं लेखकभ्रमजं द्रष्टव्यम् । यद्वा न केवलं कल्पजैर्भगणकुदिनाधिमासा-वमादिभिर्ग्रहाहर्गणाद्यानयने विकलाशेषादेस्तदानयनं किं तु युगजैरपि कुदिनाद्यैस्तत्साधने तदुत्पन्नाद्विकलाशेषाद्युगजभाज्य भाजकेभ्योऽपि तत्साधनं भवतीति सूचनाय युगावमानीत्युक्तम् । H.T. Colebrooke makes note of this view of Kṛṣṇa in his book : "*Yuga* is here an error of the transcriber for *calpa* ; or has been introduced by the author to intimate, that the method is not restricted to time calcualted by the *calpa*, but also applicable when the calculation is by the *yuga* or any other astronomical period."[7]

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6. Refer H.T. Colebrooke, *op.cit.*, p. 160, fn. 4 : "This second half of the stanza is not inserted in the *Lilāvatī.* Crīshna, the commentator of the *Vija-Ganita,* notices with censure a variation in the reading of the text: 'Those deduced from a negative dividend, being treated in the same manner, become the results of a negative divisor.' "

7. *ibid.*, p. 168, fn. 3

6)An example from the chapter on *Ekavarṇa samīkaraṇam* reads: असमानसमच्छेदान्राशींस्तांश्चतुरो वद । While discussing the problem, Kṛṣṇa makes note of an alternative opening of the verse as असमानसमप्रज्ञ. He prefers the second reading to the first (found in all edited texts and also here), and rejects the term *samaccheda* (having like denominators), as it is not necessary, and it is not made a condition of the problem, though it rises out of the solution (*BP.* p. 179) : असमानसमप्रज्ञेति पाठे तु हे असमप्रज्ञ, निरुपमबुद्धे असमांस्तांश्चतुरो राशीन्वदेति योजनीयम् । प्रथमपाठस्त्वसाधुरिति प्रतिभाति । न हि समच्छेदत्वपुरस्कारेणोदाहरणमिह साध्यते । किन्तु समच्छेदत्वं संपातायातम् ।

7) Kṛṣṇa quotes not less than three versions of the last line of *BG.* v. 154 in the beginning of the section on *Anekavarṇasamīkaraṇa* (*BP.* pp. 206-07) :

1) भूयः कार्यः कुट्टकोऽत्रान्त्यवर्णम् ।
2) भूयः कार्यः कुट्टकादन्यवर्णः ।
3) भूयः कार्यः कुट्टकादन्यवर्णः तेनोत्थाप्योत्थापयेदन्तिमाद्यान् ।

To support the first *pāṭha*, Kṛṣṇa quotes Bhāskara (*BG.* p. 77) *verbatim* : "अथ यदि विलोमोत्थापने क्रियमाणे पूर्ववर्णोन्मितौ तन्मितिर्भिन्ना लभ्यते तदा कुट्टकविधिना यो गुणः सक्षेप उत्पद्यते स भाज्यवर्णस्य मानं तेनान्त्यवर्णमानेषु तं वर्णमुत्थाप्य पूर्वोन्मितिषु विलोमोत्थापनप्रकारेण अन्यवर्णमानानि . . . ।" — But if the values (got by substitution and inverse process) are fractions and not integers then *Kuṭṭaka* process should be employed again. By substitution and backward process we should get the values of x and other unknowns.[8] Hence Kṛṣṇa says that the term *antyavarṇa* is proper here.

Kṛṣṇa continues : इह हि यदि अन्यवर्णमानं भिन्नं स्यात् तदा भूयः कुट्टकादन्यवर्ण कार्य इत्युक्तेरन्यवर्णो भाजकवर्ण एव । एवं सति भाजकवर्णमानेन अन्त्यवर्णमुत्थाप्य इत्यर्थः पर्यवस्यति । नचासौ युक्तः । भाजकवर्णान्त्यवर्णयोः भेदात् । किंतु भाज्यवर्णस्यान्त्यवर्णस्य चाभेदात् भाज्यवर्णमानेनैवान्त्यवर्णोत्थापनं युक्तम् । तदेवं द्वितीय पाठो न साधुः । एवं तृतीयपाठोप्यसाधुः ।

8. *Cf. ibid.*, pp.229-30.

If the reading *anyavarṇa* is taken, then it would be mean "substitute another value". But this cannot be done since the divisor and last value of x are different. But it can be done for the dividend ; Thus it can be seen that the second reading is faulty, so also the third one.[9]

8) In the chapter on *Bhāvita*, Bhāskara explains the solutions of a linear equation through *kṣetragaṇita*. But there seems to be a lacunae in his gloss which Kṛṣṇa points out as scribal error. Hence Kṛṣṇa gives a detailed proof for the same (*BP.* p. 258) : किं तु लेखकादिदोषादुपदेशविच्छित्त्या च संप्रति सा न स्वकार्यक्षमा । अत: इयं भावितोपपत्ति: विविच्योच्यते ।

Sudhākara Dvivedi seems to endorse him fully: अत्र मूले लेखकाध्यापकाध्येतृदोषै: काचित् त्रुटिरस्ति । तदर्थं कृष्णदैवज्ञकृता नवाङ्कुराख्या बीजगणितटीका विलोक्या ।[10]

7.6. Authorities cited by Kṛṣṇa :

Kṛṣṇa while elucidating the *sūtras* of *BG,* substantiates them by quoting from Bhāskara's other works, *Līlāvatī* and *Siddhānta Śiromaṇi* and also works by various other authors. These are enlisted below :

7.6.1. Quotations from *Līlāvatī*:

Līlāvatī	Context	*BP.*
v. 71	solving specific equation using arithmetic	p. 5
v. 19	divisions of a negative number by a positive number and vice versa	p. 18
v. 25	operations of zero	p. 21
v. 46d	any number multiplied by zero becomes zero	p.26

9. *Cf. ibid.*, p. 228, fn.2.

17 10. Ed. *Bījagaṇita, op.cit.*, p. 126

v. 44	finding squares and cubes of fractions	p. 56
vv. 126,158, 160	examples of *Ekāvarna sāmikaraṇa* and *Madhyamāharaṇa*	192, 182, 182

7.6.2. Quotation of *Siddhānta Śiromaṇi*:

Madhyamādhikāra	Context	*BP.*
v. 12	explaining the benedictory verse	p. 9

Tripraśnādhyāya	Context	*BP.*
v. 93	solving specific equation using arithmetic	p. 5
v. 101-02	solving specifying equation using arithmetic	p. 5
v.100	finding two roots through *madhyamāharaṇa*	p. 188

Golādhyāya	Context	*BP.*
v. 6	explaining the benedictory verse	pp. 8-9

Chedyādhikāra	Context	*BP.*
v. 9	explaining the benedictory verse	p. 8

Praśnādhyāya	Context	*BP.*
v. 2	necessity for teaching (algebra) unknown variables	p. 7
vv. 10, 13	*samsliṣṭakuṭṭaka*	p. 129

v. 17	while solving *Sthirakuṭṭaka*	p. 122
v. 24	*kuṭṭaka-apavartana*	p. 228

7.6.3. Quotations from other authors :

a) Brahmagupta (*Br. Sp.* XII, 1.) is cited in the chapter on *Kuṭṭaka* by Kṛṣṇa (*BP.* p. 85) :

परिकर्म विशतिं च संकलिताद्यां पृथक् विजानाति ।
अष्टौ च व्यवाहारात् छायोन्तान् भवति गणकः सः ॥

b) Gaṇeśa Daivajña's commentary *Buddhavilāsini* (see **3.3**) on *Līlāvatī,* v. 243 is quoted by Kṛṣṇa (p. 86) while explaining the term *dṛḍhasamjñā.*

c) Śrīdharācārya's famous *sūtra* is quoted in the section on *Madhyamāharaṇa* in order to equate both sides to get the root (*BP.* p. 188) :

चतुराहतवर्गसमरुपैः पक्षद्वयं गुणयेत् ।
पूर्वाव्यक्तस्यकृतेः समरूपाणि क्षिपेत्तयोरेव ॥

d) Viṣṇu Daivajña, the revered *guru* of Kṛṣṇa is proudly quoted by him on two occasions. One of his *sūtras* (*BP.* p. 231) suggests a simple solution for the problem *ṣaḍaṣtaśatakāḥ* . . . ॥ Another rule of Viṣṇu Daivajña is quoted in *BP.* p. 216 in solving the problem *eko bravīti* . . . ॥ Both the rules have been cited in the section on *anekavarṇasamīkaraṇa.*

e) Īśvarakṛṣṇa's *Sāṅkhyakārīkā* is cited by Kṛṣṇa (*BP.* pp. 3-4) while explaining the benedictory verse.

7.7. Later writers on Kṛṣṇa :

a) Raṅganātha, at the end of his commentary *Gūḍhārthaprakāśaka* on *Sūryasiddhānta* extols his elder brother Kṛṣṇa as a great mathematician, who adorned the court of the Mughal emperor Jahangir.

b) The same idea has been echoed by Munīśvara in his commentary *Marīcī* on *Siddhānta Śiromaṇi*.[11]

c) Kamalākara, in his *Siddhāntatattvaviveka* observes that the rule extracted by Kṛṣṇa from Golādhyāya section of *Siddhānta Śiromaṇi* is out of context. (This has already been discussed in **6.1.2.1.**)

7.8. Modern scholars and Kṛṣṇa :

Bhāskara's *BG* had been discussed by many scholars through the centuries all over the world. By 19th century, Kṛṣṇa's *BP* drew the attention of many scholars to understand Bhāskara better.

H.T. Colebrooke seems to be the earliest of these to have made a study of Kṛṣṇa's *BP* in detail. While translating the *BG*, Colebrooke refers to all the three commentators on *BG viz.*, Sūryadāsa, Kṛṣṇa and Rāmakṛṣṇa; of them Kṛṣṇa is quoted copiously wherever necessary.

Mm. Sudhakara Dvivedi turns to Kṛṣṇa whenever there is a doubt to be clarified in the *BG* and records the fact in his foot-notes of the edition of *BG*. He extols Kṛṣṇa in his *Gaṇakataraṅgini* (p.70): अस्यां टीकायां कृष्णदैवज्ञेन बहुत्र नवीना कल्पना स्वबुद्ध्या करण्यादिमूलसाधनादौ विलिखिता । वस्तुतः प्राचीनटीकासु कृष्णदैवज्ञकृतेयं टीकैव संप्रति ज्योतिर्विद्भिः आवृता भारतवर्षे प्रधानज्योतिर्विदां गेहे शोभते ।

Jivanatha Jha in his commentary *Subodhinī* on *BG* quotes profusely from Kṛṣṇa's *BP*.

C.O. Selenius, a mathematician of 20th century, in his "Rationale of the Chakravala process of Jayadeva and Bhāskara II", says that Kṛṣṇa uses "a more sophisticated formula" to find the greater root.

11. Praśnādhyāya, v. 8.

Datta and Singh in their excellent source book, *History of Hindu Mathematics,* quote Kṛṣṇa's methods extensively while dealing with arithmetic and algebra. In fact they have not mentioned any other commentator.

7.9. Conclusion :

From the foregoing chapters, the following facts can be considered as the original contributions of Kṛṣṇa :

1. Kṛṣṇa has explained addition and substraction of positive and negative numbers by representing them on the number line **(2.1)**.

2. He explains with reason that $a \times 0 = 0 \times a = 0$ and $\frac{a}{0} = \frac{b}{0} = \infty$ **(2.2)**.

3. We learn from Kṛṣṇa why a negative number cannot have a square root **(2.1.4)**.

4. Kṛṣṇa has added a few easy methods in *kuṭṭaka,* such as solving the equation $ax + c = by$ when $a = bl + 1$ and so on **(3.5.1)**.

5. Kṛṣṇa has given a new *upapatti* for Brahmagupta's *bhāvanā* **(4.3)**.

6. Kṛṣṇa adds a brilliant rule to find *jyeṣṭha* or y in $Nx^2 + 1 = y^2$, which avoids finding square root of large numbers **(4.5.3.1)**.

7. Kṛṣṇa gives a different treatment of a few equations in *ekavarṇa samīkaraṇa* **(5.3.)**.

8. In dealing with equations with more than one unknown, Kṛṣṇa adds a simple rule given by his *guru* Viṣṇu Daivajña **(6.3)**.

9. To find a solution for the problem in the verse – *ṣaḍaṣṭaśatakāḥ* . . . Kṛṣṇa improves on Bhāskara's method (**6.1.2.2**).

10. In the section on *anekavarṇa samīkaraṇa,* Kṛṣṇa for the first time adds a couple of verses of his own as examples (**6.2**).

11. In addition, Kṛṣṇa has identified *apapāṭhas* found in the various manuscripts of *BG* available during his time, and has made suitable corrections besides correcting the scribal error found in *BG* while solving a problem in the *Bhāvita* section.

12. Kṛṣṇa has made his treatise interesting by taking recourse to different *śāstras* like Sāṅkhya, Vyākaraṇa and Chandas.

A few shortcomings as found in *BP* are noted below :

1. In the section on *Ekavarṇa samīkaraṇa,* while solving the problem in the *BG.* v. 119, Kṛṣṇa prefers one of the two solutions arrived at. He adds that he would give the reason for preferring one of the solutions in the section on *Madhyamāharaṇa.* However, the reason is not found in that section (**5.3.2**).

2. Kṛṣṇa has given only two examples of his own. Both of them lack sufficient data to arrive at the solution (**6.2**).

To sum up, Kṛṣṇa's *BP* is a good summary of Indian Algebra of his times. In the words of T.V. Radhakrishnan, the editor of *BP* ". . . the tree of Algebra was not dead, nay, was not capable of dying! . . . Sri Krishna Daivagna realised that the tree of Algebra also should have its sprouts. So he wrote this commentary and called it the Sprout of Algebra or *Beejapallavam* announcing to the people all around that this knowledge also was bound to have a better recognition

. . . Indeed at that time to explain the principles of the *Beejaganitam* without the help of a *guru* was an uphill task."

Study of commentaries like *BP*, on texts on highly technical topics is imperative in order to understand the contribution of our ancients in various branches of science.

There is still a lot of scope for study in this field. It has been found that there is a close link between the ancient Indian mathematical methods and modern ones.

The critical edition and study of such texts by scholars well versed in both Sanskrit and Science, for proper understanding and interpretation of the wisdom contained in ancient treatises, is the need of the hour.

APPENDICES

APPENDIX - I

ON PT. JIVANANDA VIDYASAGARA'S EDITION OF *BĪJAGAṆITA*

Ms. Pushpa Kumari Jain in her recent book *The Sūryaprakāśa of Sūryadāsa* (Vol. I, Oriental Institute, Vadodara, 2001), which is another commentary on Bhāskara's *BG*, says (Introduction, p.24) : "The edition of Bhāskara's *Bījagaṇita*, by Jivananda Vidyasagara, Calcutta, 1878, which we have referred to in our work, contains Bhaskara's own commentary".

On going through the above mentioned edition of *BG*, it is found that the said edition may contain corrupt readings for the following reason : On p.121 of the said edition we find the following reading : अस्यानयनार्थं व्यक्तरीत्यैव सूत्रं कृतमस्मद् गुरुचरणै: श्री विष्णुदैवज्ञै: । शेषविक्रय . . . धीमता ।। These are words that are found verbatim in the text *BP*(p.231) of Kṛṣṇa, when he cites an example of his own *guru* , Śrī Viṣṇu Daivajña . Also in the very beginning of his text *BP*, Kṛṣṇa offers his salutions to his *guru* Śrī Viṣṇu Daivajña , when he records his *Guru parampara* :

तच्छिष्यो विष्णुनामा स जयति जगतिजागरूकप्रतिष्ठ:
शिष्टानामग्रगण्य: सुभणितगणिताम्नायविद्याशरण्य: ।
यद्वक्त्रोन्मुक्ताफलविमलवचोवीचिमालागलन्तो -
र्द्वित्रा: सिद्धान्तकेशा जगति विदधतेऽज्ञेऽपि सर्वज्ञगर्वम् ।।

तस्मादधीत्य विधिवत् त्रिस्कन्धं ज्योतिषं गुरो: ।
कृष्णो दैवविदां श्रेष्ठस्तनुते बीजपल्लवम् ।।
अव्यक्तत्वादिदं बीजमित्युक्तं शास्त्रकर्तृभि: ।
तद् व्यक्तीकरणं शक्यं न विना गुर्वनुग्रहम् ।।

He praises the scholarship of his *guru* Viṣṇu Daivajña and says he could not have written *BP* without the grace of his *guru*. In another place also he refers to his *guru* (*BP*. p.216) . . . एको ब्रवीतीत्यादि . . . तदानयनमुक्तमस्मद्गुरुभि: श्रीविष्णुदैवज्ञै: । and explains his *guru's* method of solving a problem.

On the other hand, it is well known that Bhāskara who belonged to the city of Jadabida, was the son and pupil of Maheśvara. This is also borne out by Kṛṣṇa (*BP.* p.3) : अथ शाण्डिल्यगोत्र – मुनिवरवंशावतंस – जडविडनगरनिवासि – कुंभोद्भवभूषणादिकभूषण – सकलागम आचार्यवर्य – श्री महेश्वरोपाध्यायतनय –निखिलविद्यावाचस्पति – गणितविद्याचतुरानन – धरणितर श्रीभास्कराचार्यः खगणितरूपसिद्धान्तशिरोमणिं चिकीर्षुः . . . ।

Also, we come to know of the same facts from Bhāskara himself, (*Siddhānta Śiromaṇi,* Praśnādyāya, vv. 61-2) :

आसीत्सह्यकुलाचलाश्रितपुरे त्रैविद्यविद्वज्जने
नानासज्जनधाम्नि विज्जडविडे शाण्डिल्यगोत्रो द्विजः ।
श्रौतस्मार्तविचारसारचतुरो निःशेषविद्यानिधिः
साधूनामवधिर्महेश्वरकृती दैवज्ञचूडामणिः ।।

तज्जस्तच्चरणारविन्दयुगलप्राप्तप्रसादः सुधीः
मुग्धोद्बोधकरं विदग्धगणकप्रीतिप्रदं प्रस्फुटम् ।
एतद्व्यक्तसदुक्तियुक्तिबहुलं हेलावगम्यं विदाम्
सिद्धान्तग्रथनं कुबुद्धिमथनं चक्रे कविर्भास्करः ।।

Again in *BG* (v. 211) he confirms that he procured his knowledge of algebra from his father and *guru,* Maheśvara :

आसीन्महेश्वर इति प्रथितः पृथिव्यामाचार्यवर्यपदवीं विदुषां प्रयातः ।
लब्ध्वाऽवबोधकलिकां तत एव चक्रे तज्जेन बीजगणितं लघु भास्करेण ।।

Thus, the edition of *BG* by Pt. Jivananda Vidyasagara containing both the *sūtras* and the gloss of Bhāskara seems to be corrupt, as it contains a portion of *BP* of Kṛṣṇa.

APPENDIX - II
OTHER *UPAPATTIS* ON *VARGA-PRAKṚTI*

Under section 4.4, four *upapattis* as given by Kṛṣṇa have been presented. Here two more *upapattis* found in the edition of *BG* by Acyutananda Jha, with the Sanskrit com. of Jivanatha Jha (Chaukambha Sanskrit Series 148, Varanasi, 2002), pp. 163-64.

i. *Upapatti* attributed to Bapudeva Sastri :

Let the *kaniṣṭha* be x and *jyeṣṭha* be y. Then

$$\sqrt{Nx^2+1} = y$$

Assume

$$y = xx_1 + 1$$

Then $$\sqrt{Nx^2+1} = xx_1 + 1$$

Squaring

$$Nx^2 + 1 = (xx_1+1)^2$$

$$= x^2x_1^2 + 1 + 2xx_1$$

i.e. $$2xx_1 = Nx^2 - x^2x_1^2$$

$$= x^2(N - x_1^2)$$

i.e. $$x = \frac{2x_1}{N - x_1^2}$$

i.e. $$y = xx_1 + 1$$

$$= \frac{2x_1 \times x_1}{N - x_1^2} + 1$$

$$= \frac{2x_1^2 + N - x_1^2}{N - x_1^2} = \frac{N + x_1^2}{N - x_1^2}$$

So *kaniṣṭha* $= \dfrac{2x_1}{N - x_1^2}$

jyeṣṭha $= \dfrac{N + x_1^2}{N - x_1^2}$

ii. Another *Upapatti*:

Let x_1 y_1 k_1 be 1, k, $k^2 - N$ respectively

Paṅkti or columns for the *bhāvanā* are as follows :

prakṛti	*kaniṣṭha*	*jyeṣṭha*	*kṣepa*
	1	k	$k^2 - N$
N			
	1	k	$k^2 - N$

By the method of *tulya-bhāvanā*

New *kaniṣṭha* $x = \dfrac{2k}{k^2 - N}$

New *jyeṣṭha* $y = \dfrac{N + k^2}{k^2 - N}$

New *kṣepa* $k = \dfrac{(k^2 - N)^2}{(k^2 - N)^2} = 1$

APPENDIX - III

EXPANSION OF $\sqrt{N}$ AS AN INFINITE CONTINUED FRACTION

The European method of solving the equation of the type $Nx^2 + 1 = y^2$ involves the expansion of $\sqrt{N}$ as an infinite continued fraction. In the example given below $\sqrt{58}$ is first expanded as a 'regular' continued fraction and then as a 'half regular' continued fraction.

1) Expansion of $\sqrt{58}$ as a 'regular' continued fraction :

Observing that the integral part of $\sqrt{58}$ is 7, we write

$$\sqrt{58} = a_1 + \frac{1}{a_2} = 7 + \frac{1}{a_2},\ a_1 = 7\ ;,$$

i.e. $\sqrt{58} - 7 = \dfrac{1}{a_2}$

So,

$$a_2 = \frac{1}{\left(\sqrt{58}-7\right)}\ \frac{\sqrt{58}+7}{\left(\sqrt{58}+7\right)} = \frac{\sqrt{58}+7}{58-49} = \frac{\sqrt{58}+7}{9} = 1 + \frac{1}{a_3}$$

say, where $\dfrac{\sqrt{58}+7}{9} - 1 = \dfrac{1}{a_3} \Rightarrow \dfrac{1}{a_3} = \dfrac{\sqrt{58}-2}{9}$

$$a_3 = \frac{9\left(\sqrt{58}+2\right)}{\left(\sqrt{58}-2\right)\left(\sqrt{58}-2\right)} = \frac{9\left(\sqrt{58}+2\right)}{54} = 1 + \frac{1}{a_4},\ \text{say}$$

where $\dfrac{\sqrt{58}+2}{6} = 1 + \dfrac{1}{a_4} \Rightarrow \dfrac{\sqrt{58}+2-6}{6} = \dfrac{1}{a_4} = \dfrac{\sqrt{58}-4}{6}$

Now $a_4 = \dfrac{6\left(\sqrt{58}+4\right)}{\left(\sqrt{58}-4\right)\left(\sqrt{58}+4\right)} = \dfrac{6\left(\sqrt{58}+4\right)}{42} = 1 + \dfrac{1}{a_5}$, say,

where $\dfrac{\sqrt{58}+4}{7} - 1 = \dfrac{\sqrt{58}-3}{7} = \dfrac{1}{a_5}$

i.e. $a_5 = \dfrac{7}{\sqrt{58}-3} = \dfrac{7\left(\sqrt{58}+3\right)}{\left(\sqrt{58}-3\right)\left(\sqrt{58}+3\right)} = \dfrac{7\left(\sqrt{58}+3\right)}{49} = 1 + \dfrac{1}{a_6}$, say

where $\dfrac{\sqrt{58}+3}{7} - 1 = \dfrac{1}{a_6} = \dfrac{\sqrt{58}-4}{7}$

i.e. $a_6 = \dfrac{7}{\left(\sqrt{58}-4\right)} = \dfrac{7\left(\sqrt{58}+4\right)}{\left(\sqrt{58}+4\right)\left(\sqrt{58}-4\right)} = \dfrac{7\left(\sqrt{58}+4\right)}{42} = 1 + \dfrac{1}{a_7}$, say

where $\dfrac{\sqrt{58}+4-6}{6} = \dfrac{1}{a_7} = \dfrac{\sqrt{58}-2}{6}$

i.e. $a_7 = \dfrac{6\left(\sqrt{58}+2\right)}{\left(\sqrt{58}-2\right)\left(\sqrt{58}+2\right)} \; \dfrac{6\left(\sqrt{58}+2\right)}{54} = 1 + \dfrac{1}{a_8}$, say, where

$\dfrac{1}{a_8} = \dfrac{\sqrt{58}+2-9}{9} = \dfrac{\sqrt{58}-7}{9}$

i.e. $a_8 = \dfrac{9\left(\sqrt{58}+7\right)}{\left(\sqrt{58}+7\right)\left(\sqrt{58}-7\right)} = \dfrac{9\left(\sqrt{58}+7\right)}{9} = 9a_2$ where integral part is 14.

Thus $\sqrt{58} = \left\{7, \overline{1, 1, 1, 1, 1, 1, 14}\right\}$

2) 'Half regular' expansion of $\sqrt{58}$ as continued fraction :

$\sqrt{58} = a_1 + \frac{1}{a_2} \Rightarrow 8 - \sqrt{58} = \frac{1}{a_2}$ (8 being the nearest integer larger than $\sqrt{58}$

$$\text{i.e. } a_2 = \frac{1}{8-\sqrt{58}} = \frac{8+\sqrt{58}}{64-58} = \frac{8+\sqrt{58}}{6} = 2+\frac{1}{a_3}$$

$$\text{so, } \frac{1}{a_3} = \frac{8+\sqrt{58}}{6} - 2 = \frac{\sqrt{58}-4}{6}$$

$$\text{or } a_3 = \frac{6}{\sqrt{58}-4} \quad \frac{\sqrt{58}+4}{\sqrt{58}+4} = \frac{6\left(\sqrt{58}+4\right)}{42} = 1+\frac{1}{a_4}, \text{ say}$$

$$\text{where } \frac{\sqrt{58}+4}{7} - 1 = \frac{1}{a_4} = \frac{\sqrt{58}+4-7}{7} = \frac{\sqrt{58}-3}{7}$$

$$\text{i.e. } a_4 = \frac{7\left(\sqrt{58}+3\right)}{\left(\sqrt{58}-3\right)\left(\sqrt{58}+3\right)} = \frac{7\left(\sqrt{58}+4\right)}{49} = 1+\frac{1}{a_5}, \text{ say}$$

$$\frac{1}{a_5} = \frac{\sqrt{58}+3}{7} - 7 = \frac{\sqrt{58}-4}{7}$$

$$\text{i.e. } a_5 = \frac{7\left(\sqrt{58}+4\right)}{\left(\sqrt{58}-4\right)\left(\sqrt{58}+4\right)} = \frac{7\left(\sqrt{58}+4\right)}{42} = 1+\frac{1}{a_6}, \text{ say}$$

$$\frac{1}{a_6} = \frac{\sqrt{58}+4-6}{6} = \frac{\sqrt{58}-2}{6}$$

$$\text{i.e. } a_6 = \frac{6\left(\sqrt{58}+2\right)}{\left(\sqrt{58}-2\right)\left(\sqrt{58}+2\right)} = \frac{6\left(\sqrt{58}+2\right)}{54} = \frac{\sqrt{58}+2}{9} = 1+\frac{1}{a_7}, \text{ say}$$

where $\dfrac{1}{a_7} = \dfrac{\sqrt{58}+2-9}{9} = \dfrac{\sqrt{58}-7}{9}$

i.e. $a_7 = \dfrac{9\left(\sqrt{58}+7\right)}{\left(\sqrt{58}+7\right)\left(\sqrt{58}-7\right)} = \dfrac{9\left(\sqrt{58}+7\right)}{9}$

$\sqrt{58} = \left\{8, \overline{2, 1, 1, 1, 1, 15}\right\}$

The Appendix V is used to calculate convergents. Here

$p_i = a_i p_{i-1} + p_{i-2}\ ;\ q_i = a_i q_{i-1} + q_{i-2},\quad i = 1, 2, 3, \ldots$

In the context of the equations of the form $Nx^2 + 1 = y^2$ where N is not a perfect square so that $\sqrt{N}$ is a (quadratic) surd, the following theorem is well-known :

Theorem : The continued fraction which represents a quadratic surd is periodic.[1]

[2]The following result is also well known : If n is the period of the continued fraction for N and $\frac{p_k}{q_k}$ is the k^{th} convergent of the continued fraction, then $p_n^2 - Nq_n^2 = (-1)^n$,

Case 1 : n is even

Then equation $p_n^2 - Nq_n^2 = (-1)^n$

becomes $p_n^2 - Nq_n^2 = 1$

Therefore $x = q_n$, $y = p_n$ can be a particular solution of $Nx^2 + 1 = y^2$.

1. C.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers* (Oxford, 1954), p. 144.

2. C.D. Olds, *Continued Fractions*, California, 1961, p. 115.

Case 2 : n is odd

Then equation $p_n^2 - Nq_n^2 = (-1)^n$

becomes $p_n^2 - Nq_n^2 = -1$

showing that $x = q_n$, $y = p_n$ is a solution of $Nx^2 - 1 = y^2$. To get the solutions for the equation $Nx^2 + 1 = y^2$ we move to the second period in the expansion of $\sqrt{N}$ to the term a_n where it occurs again ; this is actually the term a_{2n} .

Then $p_{2n}^2 - Nq_{2n}^2 = (-1)^{2n} = 1$.

So $x = q_{2n}$. $y = p_{2n}$ gives us a particular solution of $Nx^2 + 1 = y^2$.

APPENDIX - IV

ANCIENT INDIAN PRESENTATION OF THE PROBLEM - षडष्टशतका: क्रीत्वा . . . ।

(In **6.1.2.2 Method 1** the same problem is presented with modern algebraic notation. The method given below is as found in *BP.* pp. 228-29).

विक्रय: या १ ; ऋणशेषलब्धि का १ ; इष्टविक्रय: ११०

धनशेषलब्धय:			का	६	रू	१̇
			का	८	रू	१̇
			का	१००	रू	१̇
धनशेषाणि	या	६	का	६६̇०	रू	११०
	या	८	का	८८̇०	रू	११०
	या	१००	का	११̇०००	रू	११०
शेषफलपणा:	या	३०	का	३३̇००	रू	५५०
	या	४०	का	४४̇००	रू	५५०
	या	५००	का	५५̇०००	रू	५५०
स्वस्वपणैर्युता	या	३०	का	३२̇९४	रू	५४९
	या	४०	का	४३̇९२	रू	५४९
	या	५००	का	५४̇९००	रू	५४९
समशोधने कृते	या	५	का	५४९		
कुट्टकेन जाते	नी	५४९	रू	०	या	१
	नी	५	रू	०	या	१
स्वमानेनोत्थाप्य	नी	३०	रू	१̇		
जाता लब्धय:	नी	४०	रू	१̇		
	नी	५००	रू	१̇		

अत्र नीलकम् एकेनैव उत्थापयेत् ।

APPENDIX - V

Calculation of p_i and q_i, $i = 1, 2, 3 \ldots 14$

I	$_{1}$1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_i			7	1	1	1	1	1	1	14	1	1	1	1	1	1
p_i	0	1	7	8	15	23	38	61	99	1447	1546	2993	4539	7532	12071	19603
q_i	1	0	1	1	2	3	5	8	13	190	203	393	596	989	1585	2574

$$p_n \parallel a_n p_{n_1 1} + p_{n_1 2}$$

$$q_n \parallel a_n q_{n_1 1} + q_{n_1 2}$$

$$c_n = \frac{p_n}{q_n}$$

APPENDIX - VI

GLOSSARY OF TECHNICAL TERMS

अग्र (*agra*)	–	remainder
अपवर्तनम् (*apavartanam*)	–	dividing by common divisor
अभिन्न (*abhinna*)	–	integer
अवलम्बकम् (*avalambakam*)	–	attitude of a triangle
अव्यक्त (*avyakta*)	–	unknown
आढ्य (*āḍhya*)	–	added to
आदि (*ādi*)	–	first term
आबाधा (*ābādhā*)	–	projection
आलापित (*ālāpita*)	–	as given in equation
इष्ट (*iṣṭa*)	–	desired or assumed (variable)
उत्थाप्य (*utthāpya*)	–	after substitution
उन्मितिः (*unmitiḥ*), मानं (*mānam*)	–	value
ऋण (*ṛṇa*), अस्व (*asva*)	–	negative (quantity)
कनिष्ठमूल (*kaniṣṭhamūla*), ह्रस्वमूल (*hrasvamūla*)	–	first variable as 'x' in $Nx^2+k = y^2$
करणी (*karaṇī*)	–	surd
कर्ण (*karṇa*)	–	hypotenuse of a right angled triangle
कुट्टक (*kuṭṭaka*)	–	pulveriser, a type of equation
कृति (*kṛtiḥ*), वर्ग (*vargaḥ*)	–	square

क्षेपक (*kṣepaka*), क्षेप (*kṣepa*)	–	augment
ख (*kha*), शून्य (*śūnya*)	–	zero
खहर (*khahara*)	–	infinity
खिल (*khila*)	–	not possible
गच्छ (*gaccha*)	–	number of terms of an arithmetic progression
गणितम् (*gaṇitam*)	–	mathematics
गुणक (*guṇaka*), गुणकार (*guṇakāra*)	–	multiplier
गुणन (*guṇana*), घात (*ghāta*)	–	multiplication
गुणनफ़ल (*guṇaphala*)	–	product
गुण्य (*guṇya*)	–	multiplicand
घन (*ghana*)	–	cube
चतुरस्रक्षेत्र (*caturasrakṣetra*)	–	a quadrilateral
चय (*caya*)	–	common difference
छेदगम (*chedagama*)	–	disappearance of denominator
ज्येष्ठमूल (*jyeṣṭhamūla*)	–	second variable as 'y' in $Nx^2+k=y^2$
तक्षण (*takṣaṇa*)	–	a type of subtraction
त्र्यस्रक्षेत्र (*tryasrakṣetra*)	–	triangle
दो: (*doḥ*), कोटि (*koṭi*)	–	sides of triangle
धनं (*dhanam*), स्व (*sva*)	–	positive (quantity)
निरग्र (*niragra*)	–	without remainder
प्रकृति (*prakṛti*)	–	coefficient

फल (*phala*)	–	quotient, area of triangle
बीजगणितम् (*bījagaṇitam*)	–	algebra
भागहार (*bhāgahāra*)	–	division
भाज्य (*bhājya*)	–	dividend
भावना (*bhāvanā*)	–	generator, a process of multiplication
भावितम् (*bhāvitam*)	–	product of two unknown variables ; chapter dealing with the same
भिन्न (*bhinna*)	–	fraction
मध्यमाहरण (*madhyamāharaṇa*)	–	quadratic eqation; chapter on the same
मूल (*mūla*), पद (*pada*)	–	square root
यावत् तावत् (*yāvat tāvat*)	–	unknown variable
योजक (*yojaka*)	–	Number that is added
योज्य (*yojya*)	–	Number being added to
रूप (*rūpa*)	–	one, number
लब्धि (*labdhi*)	–	quotient
वज्राभ्यास (*vajrābhyāsa*)	–	a kind of cross multiplication
वध (*vadha*)	–	multiplication
वियोजक (*viyojaka*)	–	Number that is subtracted
वियोज्य (*viyojya*)	–	Number being subtracted from
विषम (*viṣama*)	–	odd
शेष (*śeṣa*)	–	remainder

संकलन (*saṅkalana*)	–	addition
संक्रमण (*saṅkramaṇa*)	–	a method of solving simultaneous equation
संशोध्यमानं (*samśodhyamānam*)	–	subtraction
सम (*sama*)	–	even
हर (*hara*)	–	divisor

APPENDIX - VII

SŪTRAS OF *BĪJAGAṆITA*

उत्पादकं यत् प्रवदन्ति बुद्धेरधिष्ठितं सत्पुरुषेण सांख्याः ।
व्यक्तस्य कृत्स्नस्य तदेकबीजम् अव्यक्तमीशं गणितं च वन्दे ।। (1)

पूर्वं प्रोक्तं व्यक्तमव्यक्तबीजं
प्रायः प्रश्ना नो विनाऽव्यक्तयुक्त्या ।
ज्ञातुं शक्या मन्दधीभिर्नितान्तं
यस्मात् तस्माद् वच्मि बीजक्रियां च ।। (2)

१. धनर्णषड्विधम् ।

योगे युतिः स्यात् क्षययोः स्वयोर्वा धनर्णयोरन्तरमेव योगः ।। (3)

रूपत्रयं रूपचतुष्टयं च क्षयं धनं वा सहितं वदाऽऽशु ।
स्वर्णं क्षयः स्वं च पृथक् पृथङ्के धनर्णयोः संकलनामवैषि ।। (4)

[1]अत्र रूपाणामव्यक्तानां चाऽऽद्याक्षराणि उपलक्षणार्थं लेख्यानि ।
तथा यानि ऋणगतानि तानि ऊर्ध्वबिन्दूनि चेति ।। (5)

एवं भिन्नेष्वपीति ।। (6)

संशोध्यभानं स्वमृणत्वमेति स्वत्वं क्षयस्तद्युतिरूक्तवच्च ।। (7)

त्रयाद् द्वयं स्वात् स्वमृणादृणं च व्यस्तं च संशोध्य वदाऽऽशु शेषम् ।। (8)

स्वयोरस्वयोः स्वं वधः स्वर्णघाते क्षयः ।। (9)

धनं धनेनर्णमृणेन निघ्नं द्वयं त्रयेण स्वमृणेन किं स्यात् ।। (10)

भागहारेऽपि चैवं निरुक्तम् ।। (11)

1. 5-6, belong to Bhāskara's *Vāsanā* on 4 above.

रूपाष्टकं रूपचतुष्टयेन धनं धनेनर्णमृणेन भक्तम् ।
ऋणं धनेन स्वमृणेन किं स्याद् द्रुतं वदेदं यदि बोबुधीषि ।। (12)

कृतिः स्वर्णयोः स्वं स्वमूले धनर्णे ।
न मूलं क्षयस्यास्ति तस्याकृतित्वात् ।। (13)

धनस्य रूपत्रितयस्य वर्गं क्षयस्य च ब्रूहि सखे ममाऽऽशु ।। (14)

धनात्मकानामधनात्मकानां मूलं नवानां च पृथक् वदाऽऽशु ।। (15)

२. शून्यषड्विधम् ।

खयोगे वियोगे धनर्णं तथैव च्युतं शून्यतस्तद्विपर्यासमेति ।। (16)

रूपत्रयं स्वं क्षयगं च खं च ।
किं स्यात् खयुक्तं वद खच्युतं च ।। (17)

वधादौ वियत् खस्य खं खेन घाते ।
खहारो भवेत् खेन भक्तश्च राशिः ।। (18)

द्विघ्नं त्रिहृत्खं खहृतं त्रयं च शून्यस्य वर्गं वद मे पदं च ।। (19)

अस्मिन् विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु ।
बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत् ।। (20)

३. वर्णषड्विधम् ।

यावत्तावत्कालको नीलकोऽन्यो वर्णः पीतो लोहितश्चैतदाद्याः ।
अव्यक्तानां कल्पिता मानसंज्ञास्तत्संख्यानां कर्तुमाचार्यवर्यैः ।। (21)

योगोऽन्तरं तेषु समानजात्योर्विभिन्नजात्योश्च पृथक् स्थितिश्च ।। (22)

स्वमव्यक्तमेकं सखे सैकरूपं धनाव्यक्तयुग्मं विरूपाष्टकं च ।
युतौ पक्षयोरेतयोः किं धनर्णे विपर्यस्य चैक्ये भवेत् किं वदाऽऽशु ।। (23)

धनाव्यक्तवर्गत्रयं सत्रिरूपं क्षयाव्यक्तयुग्मेन युक्तं च किं स्यात् ।। (24)

धनाव्यक्तयुग्माद् ऋणाव्यक्तषट्कं स्वरूपाष्टकं प्रोक्तशेषं वदाऽऽशु ।। (25)

स्याद् रूपवर्णाभिहतौ तु वर्णो द्वित्र्यादिकानां समजातिकानाम् ।
वधे तु तद्वर्गघनादयः स्युस्तद्भावितं चासमजातिघाते ।
भागादिकं रूपवदेव शेषं व्यक्ते यदुक्तं गणिते तदत्र ।। (26)

गुण्यः पृथग्गुणकखण्डसमो निवेश्यस्तैः खण्डकैः क्रमहतः सहितो यथोक्त्या ।
अव्यक्तवर्गकरणीगुणनासु चिन्त्यो व्यक्तोक्तखण्डगुणनाविधिरेवमत्र ।। (27)

यावत्तावत्पञ्चकं व्येकरूपं यावत्तावद्भिस्त्रिभिः सद्विरूपैः ।
संगुण्य द्राग् ब्रूहि गुण्यं गुणं वा व्यस्तं स्वर्णं कल्पयित्वा च विद्वन् ।। (28)

भाज्याच्छेदः शुध्यति प्रच्युतः सन् स्वेषु स्वेषु स्थानकेषु क्रमेण ।
यैर्यैर्वर्णैः संगुणो यैश्च रूपैर्भागहारे लब्धयस्ताःस्युरत्र ।। (29)

रूपैः षड्भिर्वर्जितानां चतुर्णामव्यक्तानां ब्रूहि वर्गं सखे मे ।। (30)

कृतिभ्य आदाय पदानि तेषां द्वयोर्द्वयोश्चाभिहतिं द्विनिघ्नीम् ।
शेषात् त्यजेद्रूपपदं गृहीत्वा चेत्सन्ति रूपाणि तथैव शेषम् ।। (31)

यावत्तावत्कालकनीलकवर्णास्त्रिपञ्चसप्तधनम् ।
द्वित्र्येकमितैः क्षयगैः सहिता रहिताः कति स्युस्तैः ।। (32)

यावत्तावत्त्रयमृणमृणं कालकौ नीलकः स्वं
रूपेणाऽऽढ्या द्विगुणितमितैस्तैस्तु तैरेव निघ्नाः ।
किं स्यात्तेषां गुणनजफलं गुण्यभक्तं च किं स्याद्
गुण्यस्याथ प्रकथय कृतिं मूलमस्याः कृतेश्च ।। (33)

४. करणीषड्विधम् ।

योगं करण्योर्महतीं प्रकल्प्य घातस्य मूलं द्विगुणं लघुं च ।
योगान्तरे रूपवदेतयोस्ते वर्गेण वर्गं गुणयेद् भजेच्च ।। (34)

लघ्व्या हृतायास्तु पदं महत्या सैकं निरेकं स्वहतं लघुघ्नम् ।
योगान्तरे स्तः क्रमशस्तयोर्वा पृथक् स्थितिः स्याद् यदि नास्ति मूलम् ॥ (35)

द्विकाष्टमित्योस्त्रिभसंख्ययोश्च योगान्तरे ब्रूहि सखे करण्योः ।
त्रिसप्तमित्योश्च चिरं विचिन्त्य चेत् षड्विधं वेत्सि सखे करण्याः ॥ (36)

द्वित्र्यष्टसंख्यागुणकः करण्योर्गुण्यस्त्रिसंख्या च सपञ्चरूपा ।
वधं प्रचक्ष्वाऽऽशु विपञ्चरूपे गुणेऽथवा त्र्यर्कमिते करण्यौ ॥ (37)

क्षयो भवेच्च क्षयरूपवर्गश्चेत्साध्यतेऽसौ करणीत्वहेतोः ।
ऋणात्मिकायाश्च तथा करण्या मूलं क्षयो रूपविधानहेतोः ॥ (38)

धनर्णताव्यत्ययमीप्सितायाश्छेदे करण्या असकृद्विधाय ।
तादृक्छिदा भाज्यहरौ निहन्यादेकैव यावत् करणी हरे स्यात् ॥ (39)

भाज्यास्तया भाज्यगताः करण्यो लब्धाः करण्यो यदि योगजाः स्युः ।
विश्लेषसूत्रेण पृथक् च कार्या यथा तथा प्रष्टुरभीप्सिताः स्युः ॥ (40)

वर्गेण योगकरणी विहृता विशुद्ध्येत् खण्डानि तत्कृतिपदस्य यथेप्सितानि ।
कृत्वा तदीयकृतयः खलु पूर्वलब्ध्या क्षुण्णा भवन्ति पृथगेवमिमाः करण्यः ॥ (41)

द्विकत्रिपञ्चप्रमिताः करण्यस्तासां कृतिं द्वित्रिकसंख्ययोश्च ।
षट्पञ्चकद्वित्रिकसंमितानां पृथक् पृथङ्मे कथयाऽऽशु विद्वन् ॥
अष्टादशाष्टद्विकसंमितानां कृती कृतीनां च सखे पदानि ॥ (42)

वर्गे करण्या यदि वा करण्योस्तुल्यानि रूपाण्यथवा बहूनाम्
विशोधयेद् रूपकृतेः पदेन शेषस्य रूपाणि युतोनितानि ॥ (43)

पृथक् तदर्धे करणीद्वयं स्यान्मूलेऽथ बह्वी करणी तयोर्या ।
रूपाणि तान्येवमतोऽपि भूयः शेषाः करण्यो यदि सन्ति वर्गे ॥ (44)

ऋणात्मिका चेत् करणी कृतौ स्याद् धनात्मिका तां परिकल्प्य साध्ये ।
मूले करण्यावनयोरभीष्टा क्षयात्मिकैका सुधियाऽवगम्या ॥ (45)

त्रिसप्तमित्योर्वद मे करण्योर्विश्लेषवर्गं कृतितः पदं च ।
द्विकत्रिपञ्चप्रमिताः करण्यः स्वस्वर्णगा व्यस्तधनर्णगा वा ।
तासां कृतिं ब्रूहि कृतेः पदं च चेत् षड्विधं वेत्सि सखे करण्याः ।। (46)

एकादिसंकलितमितकरणीखण्डानि वर्गराशौ स्युः ।
वर्गे करणीत्रितये करणीद्वितयस्य तुल्यरूपाणि ।। (47)

करणीषट्के तिसृणां दशसु चतसृणां तिथिषु च पञ्चानाम् ।
रूपकृतेः प्रोज्झ्यपदं ग्राह्यं चेदन्यथा न सत् क्वापि ।। (48)

उत्पत्स्यमानयैवं मूलकरण्याऽल्पया चतुर्गुणया ।
यासामपवर्तः स्याद्रूपकृतेस्ता विशोध्याः स्युः ।। (49)

अपवर्ते या लब्धा मूलकरण्यो भवन्ति ताश्चापि ।
शेषविधिना न यदि ता भवन्ति मूलं तदा तदसत् ।। (50)

वर्गे यत्र करण्यो दन्तैः सिद्धैर्गजैर्मिता विद्वन् ।
रूपैर्दशभिरूपेताः किं मूलं ब्रूहि तस्य स्यात् ।। (51)

वर्गे यत्र करण्यास्तिथिविश्वहुताशनैश्चतुर्गुणितैः ।
तुल्या दशरूपाढ्याः किं मूलं ब्रूहि तस्य स्यात् ।। (52)

अष्टौ षट् पञ्चाशत् षष्टिः करणीत्रयं कृतौ यत्र ।
रूपैर्दशभिरुपेतं किं मूलं ब्रूहि तस्य स्यात् ।। (53)

चतुर्गुणाः सूर्यतिथीषु रुद्रनागर्तवो यत्र कृतौ करण्यः ।
सविश्वरूपा वद तत्पदं ते यद्यस्ति बीजे पटुताभिमानः ।। (54)

चत्वारिंशदशीतिद्विशतीतुल्याः करण्यश्चेत् ।
सप्तदशरूपयुक्तास्तत्र कृतौ किं पदं ब्रूहि ।। (55)

५. कुट्टकविवरणम् ।

भाज्यो हारः क्षेपकश्चापवर्त्यः केनाप्यादौ संभवे कुट्टकार्थम् ।
येन छिन्नौ भाज्यहारौ न तेन क्षेपश्चैतद्दुष्टमुद्दिष्टमेव ।। (56)

परस्परं भाजितयोर्ययोर्यः शेषस्तयोः स्यादपवर्तनं सः ।
तेनापवर्तेन विभाजितौ यौ तौ भाज्यहारौ दृढसंज्ञकौ स्तः ।। (57)

मिथौ भजेत्तौ दृढभाज्यहारौ यावद्विभाज्ये भवतीह रूपम् ।
फलान्यधोऽधस्तदधो निवेश्यः क्षेपस्तथाऽन्ते खमुपान्तिमेन ।। (58)

स्वोर्ध्वे हतेऽन्त्येन युते तदन्त्यं त्यजेन्मुहुः स्यादिति राशियुग्मम् ।
ऊर्ध्वो विभाज्येन दृढेन तष्टः फलं गुणः स्यादपरो हरेण ।। (59)

एवं तदेवात्र यदा समास्ताः स्युर्लब्धयश्चेद्विषमास्तदानीम् ।
यथागतौ लब्धिगुणौ विशोध्यौ स्वतक्षणाच्छेषमितो तु तौ स्तः ।। (60)

भवति कुट्टविधेर्युतिभाज्ययोः समपवर्तितयोरपि वा गुणः ।
भवति यो युति भाजकयोः पुनः स च भवेदपवर्तनसंगुणः ।। (61)

योगजे तक्षणाच्छुद्धे गुणाप्ती स्तो वियोगजे ।
धनभाज्योद्भवे तद्वद्भवेतामृणभाज्यजे ।। (62)

गुणलब्ध्योः समं ग्राह्यं धीमता तक्षणे फलम् ।। (63)

हरतष्टे धनक्षेपे गुणलब्धी तु पूर्ववत् ।
क्षेपतक्षणलाभाढ्या लब्धिः शुद्धौ तु वर्जिता ।। (64)

अथवा भागहारेण तष्टयोः क्षेपभाज्ययोः ।
गुणः प्राग्वत् ततो लब्धिर्भाज्याद्धतयुतोद्धृतात् ।। (65)

क्षेपाभावोऽथवा यत्र क्षेपः शुध्येद्धरोद्धृतः ।
ज्ञेयः शून्यं गुणस्तत्र क्षेपो हरहृतः फलम् ।। (66)

इष्टाहतस्वस्वहरेण युक्ते ते वा भवेतां बहुधा गुणाप्ती ।। (67)

एकविंशतियुतं शतद्वयं यद्गुणं गणक पञ्चषष्ठियुक् ।
पञ्चवर्जितशतद्वयोद्धृतं शुद्धिमेति गुणकं वदाऽऽशु तम् ।। (68)

शतं हतं येन युतं नवत्या विवर्जितं वा विहृतं त्रिषष्ठ्या ।
निरग्रकं स्याद्वद मे गुणं तं स्पष्टं पटीयान् यदि कुट्टकेऽसि ।। (69)

यद्गुणा क्षयगषष्ठिरन्विता वर्जिता च यदि वा त्रिभिस्ततः ।
स्यात् त्रयोदशहृता निरग्रका तं गुणं गणक मे पृथग्वद ।। (70)

अष्टादश गुणाः केन दशाढ्या वा दशोनिताः ।
शुद्धं भागं प्रयच्छन्ति क्षयगैकादशोद्धृताः ।। (71)

येन संगुणिताः पञ्च त्रयोविंशतिसंयुताः ।
वर्जिता वा त्रिभिर्भक्ता निरग्राः स्युः स को गुणः ।। (72)

येन पञ्चगुणिताः खसंयुताः पञ्चषष्ठिसहिताश्च तेऽथवा ।
स्युस्त्रयोदश हृता निरग्रकास्तं गुणं गणक कीर्तयाऽऽशु मे ।। (73)

क्षेपं विशुद्धिं परिकल्प्य रूपं पृथक्तयोर्ये गुणकारलब्धी ।
अभीप्सितक्षेपविशुद्धिनिघ्न्यौ स्वहारतष्टे भवतस्तयोस्ते ।। (74)

कल्प्याऽथ शुद्धिर्विकलावशेषं षष्ठिश्च भाज्यः कुदिनानि हारः ।
तज्जं फलं स्युर्विकला गुणस्तु लिप्ताग्रमस्माच्च कलालवाग्रम् ।
एवं तदूर्ध्वं च तथाऽधिमासावमाग्रकाभ्यो दिवसा रवीन्द्वोः ।। (75)

एको हरश्चेद्गुणकौ विभिन्नौ तदा गुणैक्यं परिकल्प्य भाज्यम् ।
अग्रैक्यमग्रं कृत उक्तवद्यः संश्लिष्टसंज्ञः स्फुटकुट्टकोऽसौ ।। (76)

कः पञ्चनिघ्नो विहृतस्त्रिषष्ठ्या सप्तावशेषोऽथ स एव राशिः ।
दशाहतः स्याद्विहृतस्त्रिषष्ठ्या चतुर्दशाग्रो वद राशिमेनम् ।। (77)

६. वर्गप्रकृतिः ।

इष्टं ह्रस्वं तस्य वर्गः प्रकृत्या क्षुण्णो युक्तो वर्जितो वा स येन ।
मूलं दद्यात् क्षेपकं तं धनर्णं मूलं तच्च ज्येष्ठमूलं वदन्ति ॥ (78)

ह्रस्वज्येष्ठक्षेपकान् न्यस्य तेषां तानन्यान्वाऽधो निवेश्य क्रमेण ।
साध्यान्येभ्यो भावनाभिर्बहूनि मूलान्येषां भावना प्रोच्यतेऽतः ॥ (79)

वज्राभ्यासौ ज्येष्ठलघ्वोस्तदैक्यं ह्रस्वं लघ्वोराहतिश्च प्रकृत्या ।
क्षुण्णा ज्येष्ठाभ्यासयुग्ज्येष्ठमूलं तत्राभ्यासः क्षेपयोः क्षेपकः स्यात् ॥ (80)

ह्रस्वं वज्राभ्यासयोरन्तरं वा लघ्वोर्घातो यः प्रकृत्या विनिघ्नः ।
घातो यश्च ज्येष्ठयोस्तद्वियोगो ज्येष्ठं क्षेपोऽत्रापि च क्षेपघातः ॥ (81)

इष्टवर्गहृतः क्षेपः क्षेपः स्यादिष्टभाजिते ।
मूले ते स्तोऽथ वा क्षेपः क्षुण्णः क्षुण्णे तदा पदे ॥ (82)

इष्टवर्गप्रकृत्योर्यद्विवरं तेन वा भजेत् ।
द्विघ्नमिष्टं कनिष्ठं तत्पदं स्यादेकसंयुतौ ।
ततो ज्येष्ठमिहानन्त्यं भावनातस्तथेष्टतः ॥ (83)

को वर्गोऽष्टहृतः सैकः कृतिः स्याद्गणकोच्यताम् ।
एकादशगुणः को वा वर्गः सैकः कृतिः सखे ॥ (84)

ह्रस्वज्येष्ठपदक्षेपान् भाज्यप्रक्षेपभाजकान् ।
कृत्वा कल्प्यो गुणस्तत्र तथा प्रकृतितश्च्युते ॥ (85)

गुणवर्गे प्रकृत्योनेऽथवाऽल्पं शेषकं यथा ।
तत्तु क्षेपहृतं क्षेपो व्यस्तः प्रकृतितश्च्युते ॥ (86)

गुणलब्धिः पदं ह्रस्वं ततो ज्येष्ठमतोऽसकृत् ।
त्यक्त्वा पूर्वपदक्षेपाश्चक्रवालम् इदं जगुः । (87)

चतुर्द्व्येकयुतावेवमभिन्ने भवतः पदे ।
चतुर्द्विक्षेपमूलाभ्यां रूपक्षेपार्थभावना ॥ (88)

का सप्तषष्टिगुणिता कृतिरेकयुक्ता का चैक षष्टिनिहता च सखे सरूपा ।
स्यान् मूलदा यदि कृतिप्रकृतिर्नितान्तं त्वच्चेतसि प्रवद तात ततालतावत् ॥ (89)

रूपशुद्धौ खिलोद्दिष्टं वर्गयोगो गुणो न चेत् ॥ (90)

अखिले कृतिमूलाभ्यां द्विधा रूपं विभाजितम् ।
द्विधा ह्रस्वपदं ज्येष्ठं ततो रूपविशोधने ।
पूर्ववद्वा प्रसाध्येते पदे रूपविशोधने ॥ (91)

त्रयोदशगुणो वर्गो निरेकः कः कृतिर्भवेत् ।
को वाऽष्टगुणितो वर्गो निरेको मूलदो वद ॥ (92)

को वर्गः षड्गुणस्त्र्याढ्यो द्वादशाढ्योऽथवा कृतिः ।
युतो वा पञ्चसप्तत्या त्रिशत्या वा कृतिर्भवेत् ॥ (93)

स्वबुद्ध्यैव पदे ज्ञेये बहुक्षेपविशोधने ।
तयोर्भावनयाऽऽनन्त्यं रूपक्षेपपदोत्थया ॥ (94)

वर्गच्छिन्ने गुणे ह्रस्वं तत्पदेन विभाजयेत् ॥ (95)

द्वात्रिंशद् गुणितो वर्गः कः सैको मूलदो वद ॥ (96)

इष्टभक्तो द्विधा क्षेप इष्टोनाढ्यो दलीकृतः ।
गुणमूलहृतश्चाऽऽद्यो ह्रस्वज्येष्ठे क्रमात्पदे ॥ (97)

का कृतिर्नवभिः क्षुण्णा द्विपञ्चाशद्युता कृतिः ।
को वा चतुर्गुणो वर्गस्त्रयस्त्रिंशद्युता कृतिः ॥ (98)

त्रयोदशगुणो वर्गः कस्त्रयोदशवर्जितः ।
त्रयोदशयुतो वा स्याद्वर्ग एव निगद्यताम् ॥ (99)

ऋणगैः पञ्चभिः क्षुण्णः को वर्गः सैकविंशतिः ।
वर्गः स्याद्वद चेद्वेत्सि क्षयगप्रकृतौ विधिम् ॥ (100)

उक्तं बीजोपयोगीदं संक्षिप्तं गणितं किल ।
अतो बीजं प्रवक्ष्यामि गणकानन्दकारकम् ॥ (101)

७. एकवर्णसमीकरणम् ।

यावत्तावत्कल्प्यमव्यक्तराशेर्मानं तस्मिन्कुर्वतोद्दिष्टमेव ।
तुल्यौ पक्षौ साधनीयौ प्रयत्नात्त्यक्त्वा क्षिप्त्वा वाऽपि संगुण्य भक्त्वा ॥ (102)

एकाव्यक्तं शोधयेदन्यपक्षाद्रूपाण्यन्यस्येतरस्माच्च पक्षात् ।
शेषाव्यक्ते नोद्धरेद्रूपशेषं व्यक्तं मानं जायतेऽव्यक्तराशेः ॥ (103)

अव्यक्तानां द्व्यादिकानामपीह यावत्तावद्द्व्यादिनिघ्नं हृतं वा ।
युक्तोनं वा कल्पयेदात्मबुद्ध्या मानं क्वापि व्यक्तमेवं विदित्वा ॥ (104)

एकस्य रूपत्रिशती षडश्वा अश्वा दशान्यस्य तु तुल्यमौल्याः ।
ऋणं तथा रूपशतं च यस्य तौ तुल्यवित्तौ च किमश्वमौल्यम् ॥ (105)

यदाद्यवित्तस्य दलं द्वियुक्तं तत्तुल्यवित्तो यदि वा द्वितीयः ।
आद्यो धनेन त्रिगुणोऽन्यतो वा पृथक् पृथङ् मे वद वाजिमौल्यम् ॥ (106)

माणिक्यामलनीलमौक्तिकमितिः पञ्चाष्टसप्तक्रमा-
देकस्यान्यतरस्य सप्तनवषट् तद्रत्नसंख्या सखे ।
रूपाणां नवतिर्द्विषष्टिरनयोस्तौ तुल्यवित्तौ तथा
बीजज्ञ प्रतिरत्नजानि सुमते मौल्यानि शीघ्रं वद ॥ (107)

एको ब्रवीति मम देहि शतं धनेन त्वत्तो भवामि हि सखे द्विगुणस्ततोन्यः ।
ब्रूते दशार्पयसि चेन्मम षड्गुणोऽहं त्वत्तस्तयोर्वद धने मम किं प्रमाणे ॥ (108)

माणिक्याष्टकमिन्द्रनीलदशकं मुक्ताफलानां शतं
यत्ते कर्णविभूषणे समधनं क्रीतं त्वदर्थे मया ।

19

तद्रत्नत्रयमौल्यसंयुतिमितिस्त्र्यूनं शतार्धं प्रिये
मौल्यं ब्रूहि पृथग्यदीह गणिते कल्पाऽसि कल्याणिनि ।। (109)

पञ्चांशोऽलिकुलात्कदम्बमगमत्त्र्यंशः शिलीन्ध्रं तयो-
र्विश्लेषस्त्रिगुणो मृगाक्षि कुटजं दोलायमानोऽपरः ।
कान्ते केतकमालतीपरिमलप्राप्तैककालप्रियाद्
दूताहूत इतस्ततो भ्रमति खे भृङ्गोऽलिसंख्यां वद ।। (110)

पञ्चकशतदत्तधनात्फलस्य वर्गं विशोध्य परिशिष्टम् ।
दत्तं दशकशतेन तुल्यः कालः फलं च तयोः ।। (111)

एकशतदत्तधनात्फलस्य वर्गं विशोध्य परिशिष्टम् ।
पञ्चकशतेन दत्तं तुल्यः कालः फलं च तयोः ।। (112)

माणिक्याष्टकमिन्द्रनीलदशकं मुक्ताफलानां शतं
सद्वज्राणि च पञ्च रत्नवणिजां येषां चतुर्णां धनम् ।
संगस्नेहवशेन ते निजधनाद् दत्त्वैकमेकं मिथो
जातास्तुल्यधनाः पृथग्वद सखे तद्रत्नमौल्यानि मे ।। (113)

पञ्चकशतेन दत्तं मूलं सकलान्तरं गते वर्षे ।
द्विगुणं षोडशहीनं लब्धं किं मूलमाचक्ष्व ।। (114)

यत्पञ्चकद्विकचतुष्कशतेन दत्तं खण्डैस्त्रिभिर्नवतियुक् त्रिशती धनं तत् ।
मासेषु सप्तदशपञ्चसु तुल्यमाप्तं खण्डत्रयेऽपि सकलं वद खण्डसंख्याम् ।। (115)

पुरप्रवेशे दशदो द्विसंगुणं विधाय शेषं दशभुक् च निर्गमे ।
ददौ दशैवं नगरत्रयेऽभवत्त्रिनिघ्नमाद्यं वद तत्कियद्धनम् ।। (116)

सार्धं तण्डुलमानकत्रयमहो द्रम्मेण मानाष्टकं
मुद्गानां च यदि त्रयोदशमिता एता वणिक्काकिणीः ।
आदायार्पय तण्डुलांशयुगलं मुद्गैकभागान्वितं
क्षिप्रंक्षिप्रभुजो व्रजेम हि युतः सार्थोऽग्रतो यास्यति ।। (117)

स्वार्धपञ्चांशनवमैर्युक्ताः के स्युः समास्त्रयः ।
अन्यांशद्वयहीना ये षष्टिशेषाश्च तान्वद ॥ (118)

त्रयोदश तथा पञ्च करण्यौ भुजयोर्मिती ।
भूरज्ञाताऽत्र चत्वारः फलं भूमिं वदाऽऽशु मे ॥ (119)

दशपञ्चकरण्यन्तरमेको बाहुः परश्च षट् करणी ।
भूरष्टादश करणी रूपोना लम्बमाचक्ष्व ॥ (120)

असमानसमच्छेदान्राशींस्तांश्चतुरो वद ।
यदैक्यं यद्घनैक्यं वा येषां वर्गैक्यसंमितम् ॥ (121)

त्र्यस्रक्षेत्रस्य यस्य स्यात्फलं कर्णेन संमितम् ।
दोःकोटिश्रुतिघातेन समं यस्य च तद्वद ॥ (122)

युतौ वर्गोऽन्तरे वर्गो ययोर्घाते घनो भवेत् ।
तौ राशी शीघ्रमाचक्ष्व दक्षोऽसि गणिते यदि ॥ (123)

घनैक्यं जायते वर्गो वर्गैक्यं च ययोर्घनः ।
तौ चेद्वेत्सि तदाऽहं त्वां मन्ये बीजविदां वरम् ॥ (124)

यत्र त्र्यस्रे क्षेत्रे धात्री मनुसंमिता सखे बाहू ।
एकः पञ्चदशान्यस्त्रयोदश वदावलम्बकं तत्र ॥ (125)

यदि समभुवि वेणुर्द्वित्रिपाणिप्रमाणो गणक पवनवेगादेकदेशे स भग्नः ।
भुवि नृपमितहस्तेष्वङ्ग लग्नं तदग्रम् कथय कतिषु मूलादेष भग्नः करेषु ॥ (126)

चक्रक्रौंचाकुलितसलिले क्वापि दृष्टं तडागे
तोयादूर्ध्वं कमलकलिकाग्रं वितस्तिप्रमाणम् ।
मन्दं मन्दं चलितमनिलेनाहतं हस्तयुग्मे
तस्मिन्मग्नं गणक कथय क्षिप्रमम्बुप्रमाणम् ॥ (127)

वृक्षाद्धस्तशतोच्छ्रयाच्छतयुगं वापीं कपिः कोऽप्यगा-
दुत्तीर्याथ परो द्रुतं श्रुतिपथात्प्रोड्डीय किंचिद्द्रुमात् ।

जातैवं समता तयोर्यदि गतावुड्डीयमानं कियद्
विद्वंश्चेत्सुपरिश्रमोऽस्ति गणिते क्षिप्रं तदाचक्ष्व मे ॥ (128)

पञ्चदशदशकरोच्छ्रयवेण्वोरज्ञातमध्यभूमिकयोः ।
इतरेतरमूलाग्रगसूत्रयुतेर्लम्बमानमाचक्ष्व ॥ (129)

८. मध्यमाहरणम् ।

अव्यक्तवर्गादि यदावशेषं पक्षौ तदेष्टेन निहत्य किंचित् ।
क्षेप्यं तयोर्येन पदप्रदः स्यादव्यक्तपक्षस्य पदेन भूयः ॥ (130)

व्यक्तस्य पक्षस्य समक्रियैवमव्यक्तमानं खलु लभ्यते तत् ।
न निर्वहश्चेद्घनवर्गवर्गेष्वेवं तदा ज्ञेयमिदं स्वबुद्ध्या ॥ (131)

अव्यक्तमूलर्णगरूपतोऽल्पं व्यक्तस्य पक्षस्य पदं यदि स्यात् ।
ऋणं धनं तच्च विधाय साध्यमव्यक्तमानं द्विविधं क्वचित्तु ॥ (132)

'चतुराहतवर्गसमै रूपैः पक्षद्वयं गुणयेत् ।
पूर्वाव्यक्तस्य कृतेः समरूपाणि क्षिपेत्तयोरेव' इति ॥ (133)

अलिकुलदलमूलं मालतीं यातमष्टौ निखिलनवमभागाश्चालिनी भृंङ्गमेकम् ।
निशि परिमललुब्धं पद्ममध्ये निरुद्धम् प्रतिरणति रणन्तं ब्रूहि कान्तेऽलिसंख्याम् ॥ (134)

पार्थः कर्णवधाय मार्गणगणं क्रुद्धो रणे संदधे
तस्यार्धेन निवार्य तच्छरगणं मूलैश्चतुर्भिर्हयान् ।
शल्यं षड्भिरथेषुभिस्त्रिभिरपि च्छत्रं ध्वजं कार्मुकं
चिच्छेदास्य शिरः शरेण कति ते यानर्जुनः संदधे ॥ (135)

व्येकस्य गच्छस्य दलं किलादिरादेर्दलं तत्प्रचयः फलं च ।
चयादिगच्छाभिहतिः स्वसप्तभागाधिका ब्रूहि चयादिगच्छान् ॥ (136)

कः खेन विहतो राशिः कोट्या युक्तोऽथवोनितः ।
वर्गितः स्वपदेनाऽऽढ्यः खगुणो नवतिर्भवेत् ॥ (137)

कः स्वार्धसहितो राशिः खगुणो वर्गितो युतः ।
स्वपदाभ्यां खभक्तश्च जातः पञ्चदशोच्यताम् ॥ (138)

राशिर्द्वादशनिघ्नो राशिघनाढ्यश्च कः समो यस्य ।
राशिकृतिः षड्गुणिता पञ्चत्रिंशद्युता विद्वन् ॥ (139)

को राशिर्द्विशतीक्षुण्णो राशिवर्गयुतो हतः ।
द्वाभ्यां तेनोनितो राशिवर्गवर्गोऽयुतं भवेत् ।
रूपोनं वद तं राशिं वेत्सि बीजक्रियां यदि ॥ (140)

वनान्तराले प्लवगाष्टभागः संवर्गितो वल्गति जातरागः ।
बूत्कारनाद प्रतिनादहृष्टा दृष्टा गिरौ द्वादश ते कियन्तः ॥ (141)

यूथात्पञ्चांशकस्त्र्यूनो वर्गितो गह्वरं गतः ।
दृष्टः शाखामृगः शाखामारूढो वद ते कति ॥ (142)

कर्णस्य त्रिलवेनोना द्वादशाङ्गुलशंकुभा ।
चतुर्दशाङ्गुला जाता गणक ब्रूहि तां द्रुतम् ॥ (143)

चत्वारो राशयः के ते मूलदा ये द्विसंयुताः ।
द्वयोर्द्वयोर्यथासन्नघाताश्चाष्टादशान्विताः ॥
मूलदाः सर्वमूलैक्यादेकादशयुतात्पदम् ।
त्रयोदश सखे जातं बीजज्ञ वद तान्मम ॥ (144)

राशिक्षेपाद्वधक्षेपो यद्गुणस्तत्पदोत्तरम् ।
अव्यक्तराशयः कल्प्या वर्गिताः क्षेपवर्जिताः ॥ (145)

क्षेत्रे तिथिनखैस्तुल्ये दोःकोटी तत्र का श्रुतिः ।
उपपत्तिश्च रूढस्य गणितस्यास्य कथ्यताम् ॥ (146)

दोः कोट्यन्तरवर्गेण द्विघ्नो घातः समन्वितः ।
वर्गयोगसमः स स्याद्द्वयोरव्यक्तयोर्यथा ॥ (147)

भुजात्त्र्यूनात्पदं व्येकं कोटिकर्णान्तरं सखे ।
यत्र तत्र वद क्षेत्रे दोःकोटिश्रवणान्मम ॥ (148)

वर्गयोगस्य यद्राश्योर्युतिवर्गस्य चान्तरम् ।
द्विघ्नघातसमानं स्याद्द्वयोरव्यक्तयोर्यथा ॥
चतुर्गुणस्य घातस्य युतिवर्गस्य चान्तरम् ।
राश्यन्तरकृतेस्तुल्यं द्वयोरव्यक्तयोर्यथा ॥ (149)

चत्वारिंशद्युतिर्येषां दोः कोटिश्रवसां वद ।
भुजकोटिवधो येषु शतं विंशतिसंयुतम् ॥ (150)

योगो दोःकोटिकर्णानां षट्पञ्चाशद्वधस्तथा ।
षट्शतो सप्तभिः क्षुण्णा येषां तान्मे पृथग्वद ॥ (151)

९. अनेकवर्णसमीकरणम् ।

आद्यं वर्णं शोधयेदन्यपक्षादन्यान्रूपाण्यन्यतश्चाद्यभक्ते ।
पक्षेऽन्यस्मिन्नाद्यवर्णोन्मितिः स्याद्वर्णस्यैकस्योन्मितीनां बहुत्वे ॥ (152)

समीकृतच्छेदगमे तु ताभ्यस्तदन्यवर्णोन्मितयः प्रसाध्याः ।
अन्त्योन्मितौ कुट्टविधेर्गुणाप्ती ते भाज्यतद्भाजकवर्णमाने ॥ (153)

अन्येऽपि भाज्ये यदि सन्ति वर्णास्तन्मानमिष्टं परिकल्प्य साध्ये ।
विलोमकोत्थापनतोऽन्यवर्णमानानि भिन्नं यदि मानमेवम् ।
भूयः कार्यः कुट्टकोऽत्रान्त्यवर्णं तेनोत्थाप्योत्थापयेद्व्यस्तमाद्यात् ॥ (154)

माणिक्यामलनीलमौक्तिकमितिः पञ्चाष्टसप्तक्रमा-
देकस्यान्यतरस्य सप्तनवषट् तद्रत्नसंख्या सखे ।
रूपाणां नवतिर्द्विषष्टिरनयोस्तौ तुल्यवित्तौ तथा
बीजज्ञ प्रतिरत्नजानि सुमते मौल्यानि शीघ्रं वद ॥ (155)

एको ब्रवीति मम देहि शतं धनेन त्वत्तो भवामि हि सखे द्विगुणस्ततोन्यः ।
ब्रूते दशार्पयसि चेन्मम षड्गुणोऽहं त्वत्तस्तयोर्वद धने मम किं प्रमाणे ॥ (156)

अश्वाः पञ्चगुणाङ्गमङ्गलमिता येषां चतुर्णां धना-
न्युष्ट्राश्च द्विमुनिश्रुतिक्षितिमिता अष्टद्विभूपावकाः ।

तेषामश्वतरा वृषा मुनिमहीनेत्रेन्दुसंख्या:क्रमात्
सर्वे तुल्यधनाश्च ते वद सपद्यश्वादि मौल्यानि मे ।। (157)

त्रिभि: पारावता: पञ्च पञ्चभि: सप्त सारसा: सप्तभिर्नव हंसाश्च नवभिर्बर्हिणस्त्रय: ।
द्रम्मैरवाप्यते द्रम्मशतेन शतमानय एषां पारावतादीनां विनोदार्थं महीपते: ।। (158)

षड्भक्त: पञ्चाग्र: पञ्चविभक्तो भवेच्चतुष्काग्र: ।
चतुरुद्धृतस्त्रिकाग्रो द्वयग्रस्त्रिसमुद्धृत: क: स्यात् ।। (159)

स्यु: पञ्चसप्तनवभि: क्षुण्णेषु हृतेषु केषु विंशत्या ।
रूपोत्तराणि शेषाण्यवाप्तयश्चापि शेषसमा: ।। (160)

एकाग्रो द्विहृत: क: स्याद्द्विकाग्रस्त्रिसमुद्धृत: ।
त्रिकाग्र: पञ्चभिर्भक्तस्तद्वदेव हि लब्धय: ।। (161)

कौ राशी वद पञ्चषट्कविहृतावेकद्विकाग्रौ ययो-
द्व्र्यग्रं त्र्युद्धृतमन्तरं नवहृता पञ्चाग्रका स्याद्युति: ।
घात: सप्तहृत: षडग्र इति तौ षट्काष्टकाभ्यां विना
विद्वन् कुट्टकवेदिकुञ्जर घटासंघट्टसिंहोऽसि चेत् ।। (162)

नवभि: सप्तभि: क्षुण्ण: को राशिस्त्रिंशता हृत: ।
यदग्रैक्यं फलैक्याढ्यं भवेत्षड्विंशतेर्मितम् ।। (163)

कस्त्रिसप्तनवक्षुण्णो राशिस्त्रिंशद्विभाजित: ।
यदग्रैक्यमपि त्रिंशद्धृतमेकादशाग्रकम् ।। (164)

कस्त्रयोविंशतिक्षुण्ण: षष्ट्याऽशीत्या हृत: पृथक् ।
यदग्रैक्यं शतं दृष्टं कुट्टकज्ञ वदाऽऽशु तम् ।। (165)

अत्राधिकस्य वर्णस्य भाज्यस्थस्येप्सिता मिति: ।
भागलब्धस्य नो कल्प्या क्रिया व्यभिचरेत्तथा ।। (166)

क: पञ्चगुणितो राशिस्त्रयोदशविभाजित: ।
यल्लब्धं राशिना युक्तं त्रिंशज्जातं वदाऽऽशु तम् ।। (167)

षडष्टशतकाः क्रीत्वा समार्घेण फलानि ये ।
विक्रीय च पुनः शेषमेकैकं पञ्चभिः पणैः ।।
जाताः समपणास्तेषां कः क्रयो विक्रयश्च कः ।। (168)

१०. अनेकवर्णसमीकरणान्तर्गतं मध्यमाहरणम् ।

वर्गाद्यं चेत्तुल्यशुद्धौ कृतायां पक्षस्यैकस्योक्तवद्वर्गमूलम् ।
वर्गप्रकृत्या परपक्षमूलं तयोः समीकरविधिः पुनश्च ।
वर्गप्रकृत्या विषयो न चेत्स्यात्तदाऽन्यवर्णस्य कृतेः समं तम् ।
कृत्वाऽपरं पक्षमथान्यमानं कृतिप्रकृत्याऽऽद्यमितिस्तथा च ।
वर्गप्रकृत्या विषयो यथास्यात्तथा सुधीभिर्बहुधा विचिन्त्यम् ।। (169)

बीजं मतिर्विविधवर्णसहायिनी हि मन्दावबोधविधये विबुधैर्निजाद्यैः ।
विस्तारिता गणकतामरसांशुमद्भिर्या सैव बीजगणिताह्वयतामुपैति ।। (170)

एकस्य पक्षस्य पदे गृहीते द्वितीयपक्षे यदि रूपयुक्तः ।
अव्यक्तवर्गोऽत्र कृतिप्रकृत्या साध्ये तदा ज्येष्ठकनिष्ठमूले ।। (171)

ज्येष्ठं तयोः प्रथमपक्षपदेन तुल्यं कृत्वोक्तवत्प्रथमवर्णमितिः प्रसाध्या ।
ह्रस्वं भवेत्प्रकृतिवर्णमितिः सुधीभिरेवं कृतिप्रकृतिरत्र नियोजनीया ।। (172)

को राशिर्द्विगुणो राशिवर्गैः षड्भिः समन्वितः ।
मूलदो जायते बीजगणितज्ञ वदाऽऽशु तम् ।। (173)

राशियोगकृतिर्मिश्रा राश्योर्योगघनेन च ।
द्विघ्नस्य घनयोगस्य सा तुल्या गणकोच्यताम् ।। (174)

द्वितीयपक्षे सति संभवे तु कृत्याऽपवर्त्यात्र पदे प्रसाध्ये ।
ज्येष्ठं कनिष्ठेन तथा निहन्याच्चेद्वर्गवर्गेण कृतोऽपवर्तः।।
कनिष्ठवर्गेण तदा निहन्याज्ज्येष्ठं ततः पूर्ववदेव शेषम् ।। (175)

यस्य वर्गकृतिः पञ्चगुणा वर्गशतोनिता ।
मूलदा जायते राशिं गणितज्ञ वदाऽऽशु तम् ।। (176)

कयोः स्यादन्तरे वर्गो वर्गयोगो ययोर्घनः ।
तौ राशी कथयाभिन्नौ बहुधा बीजवित्तम ॥ (177)

साव्यक्तरूपो यदि वर्णवर्गस्तदाऽन्यवर्णस्य कृतेः समं तम् ।
कृत्वा पदं तस्य तदन्यपक्षे वर्गप्रकृत्योक्तवदेव मूले ॥
कनिष्ठमाद्येन पदेन तुल्यं ज्येष्ठं द्वितीयेन समं विदध्यात् ॥ (178)

त्रिकादिद्व्युत्तरश्रेढ्यां गच्छे क्वापि च यत्फलम् ।
तदेव त्रिगुणं कस्मिन्नन्यगच्छे भवेद्वद ॥ (179)

सरूपके वर्णकृती तु यत्र तत्रेच्छयैकां प्रकृतिं प्रकल्प्य ।
शेषं ततः क्षेपकमुक्तवच्च मूले विदध्यादसकृत्समत्वे ॥ (180)

तौ राशी वद यत्कृत्योः सप्ताष्टगुणयोर्युतिः ।
मूलदा स्याद्वियोगस्तु मूलदो रूपसंयुतः ॥ (181)

घनवर्गयुतिवर्गो ययो राश्योः प्रजायते ।
समासोऽपि ययोर्वर्गस्तौ राशी शीघ्रमानय ॥ (182)

सभाविते वर्णकृती तु यत्र तन्मूलमादाय तु शेषकस्य ।
इष्टोद्धृतस्येष्टविवर्जितस्य दलेन तुल्यं हि तदेव कार्यम् ॥ (183)

ययोर्वर्गयुतिर्घातयुता मूलप्रदा भवेत् ।
तन्मूलगुणितो योगः सरूपश्चाऽऽशु तौ वद ॥ (184)

यत्स्यात्साल्पवधार्धतो घनपदं यद्वर्गयोगात्पदं
ये योगान्तरयोर्द्विकाभ्यधिकयोर्वर्गान्तरात्साष्टकात् ॥
यच्चैतत्पदपञ्चकं च मिलितं स्याद्वर्गमूलप्रदं
तौ राशी कथयाऽऽशु निश्चलमते षट्काष्टकाभ्यां विना ॥ (185)

एवं सहस्रधा गूढा मूढानां कल्पना यतः ।
क्रियया कल्पनोपायस्तदर्थमत्र कथ्यते ॥ (186)

सरूपमव्यक्तमरूपकं वा वियोगमूलं प्रथमं प्रकल्प्यम् ।
योगान्तरक्षेपकभाजिताद्यद्वर्गान्तरक्षेपकतः पदं स्यात् ॥ (187)

तेनाधिकं तत्तु वियोगमूलं स्याद्योगमूलं तु तयोस्तु वर्गौ ।
स्वक्षेपकोनौ हि वियोगयोगौ स्यातां ततः संक्रमणेन राशी ॥ (188)

राश्योर्योगवियोगकौ त्रिसहितौ वर्गौ भवेतां तयो-
र्वर्गैक्यं चतुरूनितं रवियुतं वर्गान्तरं स्यात् कृतिः ।
साल्पं घातदलं घनः पदयुतिस्तेषां द्वियुक्ता कृति-
स्तौ राशी वद कोमलामलमते षट् सप्त हित्वा परौ ॥ (189)

राश्योर्ययोः कृतिवियुती चैकेन संयुतौ वर्गौ ।
रहिते वा तौ राशी गणयित्वा कथय यदि वेत्सि ॥ (190)

यत्राव्यक्तं सरूपं हि तत्र तन्मानमानयेत् ।
सरूपस्यान्यवर्णस्य कृत्वा कृत्यादिना समम् ॥
राशिं तेन समुत्थाप्य कुर्यात्भूयोऽपरां क्रियाम् ।
सरूपेणान्यवर्णेन कृत्वा पूर्वपदं समम् ॥ (191)

यस्त्रिपञ्चगुणो राशिः पृथक्सैकः कृतिर्भवेत् ।
वदं तं बीजमन्येऽसि मध्यमाहरणे पटुः ॥ (192)

को राशिस्त्रिभिरभ्यस्तः सरूपो जायते घनः ।
घनमूलं कृतीभूतं त्र्यभ्यस्तं कृतिरेकयुक् ॥ (193)

वर्गान्तरं कयो राश्योः पृथग्द्वित्रिगुणं त्रियुक् ।
वर्गौ स्यातां वद क्षिप्रं षट्कपञ्चकयोरिव ॥ (194)

क्वचिदादेः क्वचिन्मध्यात्क्वचिदन्यात्क्रिया बुधैः ।
आरभ्यते यथा लघ्वी निर्वहेच्च यथा तथा ॥ (195)

वर्गादिर्यो हरस्तेन गुणितं यदि जायते ।
अव्यक्तं तत्र तन्मानमभिन्नं स्याद्यथा तथा ।
कल्प्योऽन्यवर्णवर्गादिस्तुल्यं शेषं यथोक्तवत् ॥ (196)

को वर्गश्चतुरूनः सन्सप्तभक्तो विशुध्यति ।
त्रिंशदूनोऽथवा कस्तं यदि वेत्सि वद द्रुतम् ॥ (197)

हरभक्ता यस्य कृतिः शुध्यति सोऽपि द्विरूपपदगुणितः ।
तेनाऽऽहतोऽन्यवर्णो रूपपदेनान्वितः कल्प्यः ॥ (198)

न यदि पदं रूपाणां क्षिपेद्धरं तेषु हारतष्टेषु ।
तावद्यावद्वर्गो भवति न चेदेवमपि खिलं तर्हि ॥ (199)

हत्वा क्षिप्त्वा च पदं यत्राऽऽद्यस्येह भवति तत्रापि ।
आलापित एव हरो रूपाणि तु शोधनानि सिद्धानि ॥ (200)

षड्भिरूनो घनः कस्य पञ्चभक्तो विशुध्यति ।
तं वदास्ति तवालं चेदभ्यासो घनकुट्टके ॥ (201)

यद्वर्गः पञ्चभिः क्षुण्णस्त्रियुक्तः षोडशोद्धृतः ।
शुद्धिमेति समाचक्ष्व दक्षोऽसि गणिते यदि ॥ (202)

११. भावितम् ।

मुक्त्वेष्टवर्णं सुधिया परेषां कल्प्यानि मानानि यथेप्सितानि ।
तथा भवेद्भावितभङ्ग एवं स्यादाद्यबीजक्रिययेष्टसिद्धिः ॥ (203)

चतुस्त्रिगुणयो राश्योः संयुतिर्द्वियुता तयोः ।
राशिघातेन तुल्या स्यात् तौ राशी वेत्सि चेद्वद ॥ (204)

चत्वारो राशयः के ते यद्योगो नखसंगुणः ।
सर्वराशिहतेस्तुल्यो भावितज्ञ निगद्यताम् ॥ (205)

यौ राशी किल या च राशि निहतिर्यौ राशिवर्गौ तथा
तेषामैक्यपदं सराशियुगलं जातं त्रयोविंशतिः ।
पञ्चाशत् त्रियुताथवा वद कियत्तद्राशियुग्मं पृथक्
कृत्वाऽभिन्नमवेहि वत्स गणकः कस्त्वत्समोऽस्ति क्षितौ ॥ (206)

भावितं पक्षतोऽभीष्टात्यक्त्वा वर्णौ सरूपकौ ।
अन्यतो भाविताङ्केन ततः पक्षौ विभज्य च ॥ (207)

वर्णाङ्काहतिरूपैक्यं भक्त्वेष्टेनेष्टतत्फले ।
एताभ्यां संयुतावूनौ कर्तव्यौ स्वेच्छया च तौ ।
वर्णाङ्कौ वर्णयोर्माने ज्ञातव्ये ते विपर्ययात् ॥ (208)

द्विगुणेन कयो राश्योर्घातेन सदृशं भवेत् ।
दशेन्द्राहतराश्यैक्यं द्व्यूनषष्टिविवर्जितम् ।। (209)

त्रिपञ्चगुणराशिभ्यां युक्तो राश्योर्वधः कयोः ।
द्विषष्टिप्रमितो जातस्तौ राशी त्वं वत्सि चेद्वद ।। (210)

१२. ग्रंथसमाप्तिः ।

आसीन्महेश्वर इति प्रथितः पृथिव्यामाचार्यवर्यपदवीं विदुषां प्रयातः ।
लब्ध्वाऽवबोधकलिकां तत एव चक्रे तज्जेन बीजगणितं लघु भास्करेण ।। (211)

ब्रह्माह्वयश्रीधरपद्मनाभबीजानि यस्मादतिविस्तृतानि ।
आदाय तत्सारमकारि नूनं सद्युक्तियुक्तं लघु शिष्यतुष्ट्यै ।। (212)

अत्रानुष्टुप्सहस्रं हि ससूत्रेद्देशके मितिः ।। (213)

क्वचित्सूत्रार्थविषयं व्याप्तिं दर्शयितुं क्वचित् ।
क्वचिच्च कल्पनाभेदं क्वचिद्युक्तिमुदाहृतम् ।
क्वचित्सूत्रार्थविषयं दर्शयितुमुदाहृतम् ।। (214)

न हि उदाहरणान्तोऽस्ति स्तोकमुक्तमिदं यतः ।। (215)

दुस्तरः स्तोकबुद्धीनां शास्त्रविस्तारवारिधिः ।
अथवा शास्त्रविस्तृत्या किं कार्यं सुधियामपि ।। (216)

उपदेशलवं शास्त्रं कुरुते धीमतो यतः ।
तत्तु प्राप्यैव विस्तारं स्वयमेवोपगच्छति ।। (217)

जले तैलं खले गुह्यं पात्रे दानं मनागपि ।
प्राज्ञे शास्त्रं स्वयं याति विस्तारं वस्तुशक्तितः ।। (218)

गणक भणिति रम्यं बालीलावगम्यं सकलगणितसारं सोपपत्तिप्रकारम् ।
इति बहुगुणयुक्तं सर्वदोषैर्विमुक्तं पठ पठ मतिवृद्ध्यै लघ्विदं प्रौढसिद्ध्यै ।। (219)

SELECT BIBLIOGRAPHY

PRIMARY TEXTS :

Bījapallavam of Kṛṣṇa Daivajña, ed. T.V. Radhakrishna Sastri, Tanjore Saraswati Mahal Series No. 78, Tanjore, 1958.

Navāṅkura, ed. Apate Dattatreya, Anandasrama Sanskrit Series 99, Pune, 1930.

Bījagaṇita with *Navāṅkura,* ed. Biharilal Vasishth, Ranbir Kendriya Samskrita Vidyapeetam, Jammu, 1982.

____________, with expository notes, examples of Mm. Sudhakara Dvivedi, ed. Pt. Muralidhara Jha, Benares Sanskrit Series 159, Benares, 1927.

____________, with Jivanatha Jha's *Subodhinī,* and Acyutananda Jha's Sanskrit and Hindi com. *Vimalā*, ed. Acyutananda Jha, Kasi Sanksrit Sereis 148, Chokhamba Sanksirt Series Office, Benares, 1949.

____________, ed. with *Sudhā,* Hindi com. and expository notes by Pt. Deva Chandra Jha, Krishnadas Academy, Varanasi, 1983.

____________, ed. Jivananda Vidyasagara, Calcutta, 1878.

SANSKRIT TEXTS :

Amarakośa, ed. with the com. of Kṣīrasvāmin by K.G. Oka, Poona, 1913.

____________, ed., A.A. Ramanathan, The Adyar Library and Research Centre, Madras, 1978.

Āryabhaṭīya of Āryabhaṭa, critically ed. by K.S. Shukla, with the com. of Bhāksara I and Someśvara, Indian National Science Academy, New Delhi, 1976.

Bījagaṇita, text with English tr. by S.K. Abhyankar, Bhaskariya Prathistana, Pune, 1980.

Brāhmasphuta Siddhānta, ed. Sudhakara Dvivedi, with com. *Vāsana* and Hindi tr. by Ram Swarup Sharma, with intro. by Satyaprakash, 4 vols., Indian Institute of Astronomy and Sanskirt Research, New Delhi, 1966.

Gaṇitasāra Saṅgraha of Mahāvīra, ed. with English tr. by M. Rangacharya, Government Press, Madras, 1912.

Golādhyāya of Bhāskara, with the com. *Marīcī* by Munīśvara, Anandasrama Sanskrit Series 122, Pt. I, 1943, Pt. II, 1952.

Līlāvatī of Bhāskara (Pts. I & II), ed. V.G. Apte, with the com. *Buddhavilāsinii* of Gaṇeśa Daivajña, Anandasrama Sanskrit Series 104, Pune, 1859.

——————, with tr. by H.T. Colebrooke, II Edition, Asian Educational Services, New Delhi, 1993.

——————, with English tr. by Sankara Patwardhan, S.A. Naimpally and S.L. Singh, Motilal Banarsidas, Delhi, 2001.

Siddhānta Śiromaṇi of Bhāskara, with his own *Vāsanabhāṣya*, Kashi Sanskrit Series No. 72, Chowkhamba Sanskrit Series Office, Benares, 1929.

——————, with English tr. by D. Arkasomayaji, Rastriya Sanskrit Vidyapeetha, Tirupati, 2000.

Siddhāntatattvaviveka of Kamalākara Bhaṭṭa, ed. with notes by Mm. Sudhakara Dvivedi and Mm. Muralidhara Jha, Chaukamba Surbharati Prakasan, Varanasi, 1991.

Sūrya-Śiddhānta, with the com. *Gūḍhārthaprakāśaka* of Raṅganātha, pub. by Babu Bhuvana Chandra Vasaka, Calcutta, 1871.

The Sūryaprakāśa of Sūryadāsa, Vol. I, ed., Ms. Pushpa Kumari Jain, with English tr., Gaekwad's Oriental Series No. 182, Oriental Institute, Vadodara, 2001.

ENGLISH TEXTS :

Bag, A.K. *Mathematics in Ancient and Medieval India,* Chowkamba Oriental Research Studies 16, Varanasi, 1979.

Bag, A.K. and Sarma, S.R. ed. *Concept of Sunya,* IGNCA and INSA, New Delhi, 2000.

Balachandra Rao, S. *Indian Mathematics and Astronomy - Some Landmarks,* Jnana Deep Publications, Bangalore, 1994.

Colebrooke, H.T. *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhaskara*, Sharada Publishing House, Delhi, 2005.

Datta, B.B. and Singh, A.N. *History of Hindu Mathematics,* Vols. I, II, Bharatiya Kala Prakasan, Delhi, 2001.

Dvivedi, Sudhakara. *Gaṇaka Taraṅgiṇiī,* The Medical Hall Press, Benares, 1892.

Gheverghese, George Joseph. *The Crest of the Peacock,* Penguin Books Ltd., 2000.

Hardy, C.H. and Wright, E.M. *An Introduction to the Theory of Numbers,* Oxford, 1954.

Hayashi, Takao. *Bakhṣhālī Manuscript* (An ancient Indian mathematical treatise), published by Egbert Forsten, Groningen, The Netherlands, 1995.

Heroor, D. Venugopal. *The History of Mathematics and Mathematicians of India,* Vidya Bharati Karnataka, Bangalore, 2006.

Ifrah, Georges. *The Universal History of Numbers,* John Wiley and Sons Inc., 2000.

Jain, Pushpa Kumari. *The Sūryaprakāśa of Sūryadāsa,* Vol. I, Oriental Institue, Vadodara, 2001.

Kaye, G.R. *Indian Mathematics,* Tharher Spink & So., Calcutta, 1915.

Olds, C.D. *Continued Fractions*, California, 1961.

Pingree, David. *Jyotiḥśāstra: Astral and Mathematical Literature,* Wiesbaden, Otto Harranowitz, 1981.

Saraswati Amma, T.A. *Geometry in Ancient and Medieval India,* Motilal Banarsidas, Delhi, 1979.

Sen, S.N. *A Bibliography on Sanskrit Works on Astronomy and Mathematics,* National Institute of Sciences of India, New Delhi, 1966.

Sen, S.N. and Bag, A.K. *The Śulbasūtras of Baudhāyana, Āpastamba, Kātyāyana and Mānava* with text and Eng. tr. and com., Indian National Science Academy, New Delhi, 1983.

Subbarayappa, B.V, and Sarma, K.V. *Indian Astronomy - Source Book,* Nehru Centre, Bombay, 1985.

Varadarajan, V.S. *Algebra in Ancient and Modern Times,* Hindustan Book Agency, New Delhi, 1997.

Venkatesha Murthy. *Kuttaka,* Rashtriya Sanskrit Vidyapeetha, Tirupati, 2004.

Weil, Andre. *Number Theory: An approach through History from Hammurapi to Legendre,* Birkhauser, 1984.

ARTICLES :

Atiyah, M. 'Mathematics as a basic science', *Current Science,* Vol. 65, No. 12, 1993.

Ayyangar, Krishnaswami, A.A. "Bhaskara and Samslishta Kuttaka" *Journal of Indian Mathematical Society, XVIII,* 1929.

__________. "New Light on Bhaskara's Chakravala or Cyclic Method of solving Indeterminate Equations of the Second Degree in two Variables", *Journal of Indian Mathematical Society 18,* 1920-30.

Bhanumurthy, T.S., "A Description of Bhāskara's Chakravala Method for the Diophantine Equation $Dx^2 - y^2 = \pm 1$ - Pt. I", *Newsletter,* Ramanujam Mathematical Society, Vol. 2, Nos.3 and 4.

Dutta, Amartya Kumar. "Brahmagupta's Lemma: The Samasabhavana", *Resonance,* 2003.

O'Connor, J.J. and Robertson, E.F. "How to solve Pell's equation", 2002, *www.history.mcs.st-andrews.ac.uk/HistTopics/Pell.html.*

Selenius, Clas Olaf. "Rationale of the Chakravala process of Jayadeva and Bhaskara II", *Historia Mathematica 2,* 1975.

20

Shukla, K.S. 'Ācārya Jayadeva, the Mathematician', *Gaṇita* 5, No. 1, 1954.

Srinivas, M.D. "The Methodology of Indian Mathematics and its contemperory relevance", *PPST Bulletin 12,* 1987.

Stedall, Jacqueline, A. "Catching Proteus - The Collaborations of Wallis and Brounker. II. Number Problems", *Notes and Records of the Royal Society of London,* Vol. 54(3), 2000.

Venkatesha Murthy, M.S. Rangachari, S. Baskaran. "Hindu Work on Linear Diophantine Equations", *Journal of Madras University*, Sec. B, 48(i) 1985.

MANUSCRIPTS :

Bījapallava of Kṛṣṇa Daivajña, TD 11523 of Tanjore Maharaja Serfoji Saraswati Mahal Library.

Sūryaprakāśa of Sūryadāsa, British Library, San. I.O. 1533a.

——————, Prajnapathasala Mandala, Wai 9777/11 - 2/551.

DICTIONARIES :

Apte, Vaman Shivram. *The Practical Sanskrit - English Dictionary,* Motilal Banarsidas, Delhi, 1965.

Monier - Williams. *A Sanskrit - English Dictionary,* Motilal Banarsidas, Delhi, 1963.

New Catalogus Catalogorum, Vols. IV,XI, University of Madras, Madras, 1963, 1983.

GENERAL INDEX

Dr. Sita Sundar Ram

B.Sc. (Mathematics), University of Madras.

M.A. (Sanskrit), Karnataka University, Dharwar.

Ph.D. (Sanskrit), University of Madras.

Has presented papers on Indian Mathematics and allied subjects in many National and International Conferences.

Has published many articles and book-reviews in leading research Journals and Magazines.

SOME IMPORTANT PUBLICATIONS OF K.S.R.I ON MATHEMATICS AND ASTRONOMY

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